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CONVERGENCE ESTIMATES FOR SEMIDISCRETE PARABOLIC EQUATION APPRO--ETC(U)
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CONVERGENCE ESTIMATES FOR SEMIDISCRETE PARABOLIC EQUATION APPROXIMATIONS

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ABSTRACT

In this paper, we study certain semidiscrete methods for approximating the solutions of initial boundary value problems, with homogeneous boundary conditions, for certain kinds of parabolic equations. These semidiscrete methods are based upon the availability of several different Galerkin-type approximation methods for the associated elliptic steady-state problem. The properties required of the spacial discretization methods are listed and estimates of the error made by the resulting semidiscrete approximations and of its time derivatives are given. In particular, estimates are given that require only weak smoothness assumptions on the initial data. Verifications of the required properties for various Galerkin-type methods are also given.

AMS(MOS) Subject Classification: Primary 65M15, Secondary 65N30

Key Words: Parabolic Equations, Semidiscrete Approximations, Galerkin Methods

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SIGNIFICANCE AND EXPLANATION

Many physical situations can be modelled by the solutions of initial boundary value problems for parabolic partial differential equations. Examples of such situations arise in the theory of heat conduction and other diffusion processes. The physical parameters involved are often dependent on the time variable.

The construction of semidiscrete approximations to the solution of such parabolic equations is studied in this paper. These approximations arise from certain spacial discretization techniques and they are governed by ordinary differential equations. Estimates for the approximation errors are given for various classes of initial data, including classes that require only weak smoothness assumptions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CONVERGENCE ESTIMATES FOR SEMIDISCRETE PARABOLIC EQUATION APPROXIMATIONS

Peter H. Sammon

I. Introduction

Let Ω be a bounded domain in d-dimensional Euclidean space with a sufficiently smooth boundary $\partial\Omega$ and an outward pointing normal $\underline{n}(x) = (n_1, \cdots, n_d)$. Let $\tau > 0$. We shall consider semidiscrete Galerkin-type approximations to the solution of the following parabolic initial boundary value problem:

$$-u_t = L(t)u \equiv -\sum_{i,j=1}^{d} D_i(a_{ij}(x,t)D_j u) + \sum_{i=1}^{d} a_{0i}(x,t)D_i u + a_0(x,t)u \quad \text{in } \Omega \times (0,\tau] \quad, \tag{1.1)(i}$$

under one of the following boundary conditions:

$$u(x,t)\Big|_{\partial\Omega} = 0 \quad \text{or} \quad \sum_{i,j=1}^{d} a_{ij}(x,t)n_i(x)D_j u(x,t)\Big|_{\partial\Omega} = 0, \quad \text{for} \quad 0 < t \le \tau \tag{1.1)(ii}$$

and with the following initial condition, where v is a known function:

$$u(x,0) = v(x) \quad \text{for} \quad x \in \Omega \quad . \tag{1.1)(iii}$$

(All functions considered in this paper will be real valued). We will put various kinds of restrictions on the initial data function v later, as well as a coercivity assumption on the coefficients of L. We will assume that $\{a_{ij}\}_{i,j=1}^{d}$, $\{a_{0i}\}_{i=1}^{d}$ and a_0 are sufficiently smooth functions on $\bar{\Omega} \times [0,\tau]$, that $a_{ij} = a_{ji}$ for $1 \le i, j \le d$ and that the matrices $[a_{ij}]_{i,j=1}^{d}$ form a uniformly positive definite family on $\bar{\Omega} \times [0,\tau]$. If the Neumann boundary conditions are under consideration, we shall also assume that

$$\begin{cases} a_{ij}(x,t) = \tilde{a}(x,t)\tilde{a}_{ij}(x) & \text{for } 1 \le i,j \le d \text{ and } (x,t) \in \bar{\Omega} \times [0,\tau] \quad , \\ a_{0i}(x,t)\Big|_{\partial\Omega} \equiv 0 & \text{for } 0 \le t \le \tau \quad , \end{cases} \tag{1.2}$$

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where $[\tilde{a}_{ij}(x)]_{i,j=1}^{d}$ is a symmetric matrix of sufficiently smooth functions on $\bar{\Omega}$ and $\tilde{a}(x,t)$ is a sufficiently smooth function on $\bar{\Omega} \times [0,\tau]$.

Suppose we consider the following elliptic boundary value problem associated with (1.1): Given $0 \leq t \leq \tau$ and a suitable function f, find a function w which satisfies $L(t)w = Lw = f$ on Ω and the appropriate homogeneous boundary conditions on $\partial\Omega$. There are many well studied techniques for finding an approximation, in a finite element space, to the solution of this problem (see the surveys in [2] or [3], for instance). Moreover, the authors of [3] have shown that it is possible to take such a technique and use it to generate a time continuous family of approximations to a solution of (1.1), at least if L has time independent coefficients. They proved that this semidiscrete approximation to (1.1) can be a good one at positive times, even if the initial data function v is not smooth on Ω or does not satisfy suitable boundary compatibility conditions. This indicates that the smoothing property of the parabolic problem can be utilized. (It is known that solutions of (1.1) are smooth in space and time for positive times, even if v is not smooth and compatible in space). They also proved uniform in time results for naturally restricted classes of initial data.

We intend to study the case of time dependent non-selfadjoint operators in this paper and show that many of the results of [3] are still valid, under similar hypotheses on the data of the problem. We will begin by setting some notation and then we will describe some results concerning the smoothing properties of the problem given by (1.1). (These results are known, but we shall give some derivations that will allow us to prove similar results for the discrete setting). We will then make some abstract hypotheses concerning finite element approximations to the associated elliptic problem, define the semidiscrete approximation and prove (optimal order) convergence results for restricted classes of smooth initial data as well as for nonsmooth initial data. It will then be shown that many of the known Galerkin-type

approximations satisfy the abstract conditions. All of these estimates will be done in a $L^2(\Omega)$-setting but we will conclude by discussing some error estimates in the maximum norm.

We refer the reader to [3] for a discussion of related work done by other authors. This work represents an extension of work done in [8] under the supervision of Professor J. H. Bramble.

We conclude this Section with an observation concerning scaling arguments. If $u(t)$ is a solution of (1.1) then $w(t) = e^{-Kt}u(t)$ is a solution of the following evolution equation:

$$w_t(t) + (L(t) + K)w(t) = 0 \quad ,$$

for any $K \in \mathbb{R}$. This relation, as well as a similar one which will hold for the semidiscrete approximation, will be used later.

We will use the symbol C to denote a generic positive constant throughout this work and we will define $\sum_{\ell=m_1}^{m_2} (\cdot) = 0$ if $m_2 < m_1$.

II. Parabolic Regularity

We will now fix some notation and discuss some of the properties of the parabolic problem defined by (1.1). We will not cite specific references for the results concerning the elliptic equation theory, but most of the statements we make may be found, for instance, in [6].

We let $W^{\ell,p} \equiv W^{\ell,p}(\Omega)$ be the usual $L^p(\Omega)$-based Sobolev spaces on Ω, where $1 \leq p \leq \infty$ and $\ell \geq 0$ is integral. We give them their usual norms, denoted by $\|\cdot\|_{\ell,p}$. We will write H^ℓ for $W^{\ell,2}$ (the $L^2(\Omega)$-based analysis will play the largest role in our work), $\|\cdot\|_\ell$ for $\|\cdot\|_{\ell,2}$ and $\|\cdot\|$ for $\|\cdot\|_0$. We will also let $\|\cdot\|$ denote the $L^2(\Omega) \to L^2(\Omega)$ operator norm. We will use the space H^1_0, which is the subspace of H^1 consisting of functions that vanish (in the sense of trace) on $\partial\Omega$, if we are considering the Dirichlet boundary conditions. We will also need to consider the space $L^2(\partial\Omega)$, whose norm will be denoted by $\|\cdot\|_{0,\partial\Omega}$ and $H^1(\partial\Omega)$, whose norm will be denoted by $\|\cdot\|_{1,\partial\Omega}$. We let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ denote the usual $L^2(\Omega)$ and $L^2(\partial\Omega)$ inner products, respectively. Finally, if X is a Banach space, we will let $C^{\ell+\epsilon}([a,b],X)$ and $C^{\ell+\epsilon}((a,b],X)$ denote the usual spaces of X-valued functions that have a Hölder continuous ℓ-th derivative, where $\ell \geq 0$ is integral and $0 \leq \epsilon < 1$ and $B(X)$ denote the usual Banach space of bounded operators on X.

We shall assume throughout this work that the initial data function $v \in L^2(\Omega)$.

We now need some concepts that pertain to elliptic equation problems that are associated with (1.1). We begin by defining some bilinear forms on $H^1 \times H^1$. Let $0 \leq t \leq \tau$ and set

$$D(t)(\cdot,\cdot) \equiv \sum_{i,j=1}^{d} (a_{ij}D_j(\cdot),D_i(\cdot)) + \sum_{i=1}^{d} (a_{0i}D_i(\cdot),(\cdot)) + (a_0(\cdot),(\cdot)),$$

$$D^*(t)(\cdot,\cdot) \equiv \sum_{i,j=1}^{d} (a_{ij}D_j(\cdot),D_i(\cdot)) - \sum_{i=1}^{d} (a_{0i}D_i(\cdot),(\cdot)) + ((a_0 - \sum_{i=1}^{d} D_i a_{0i})(\cdot),(\cdot)) \quad .$$

As usual, $D(t)$ is associated with a weak formulation of a boundary value problem for $L(t)$. $D^*(t)$ bears a similar relation to $L^*(t)$, the formal adjoint of $L(t)$, defined by $L^*(t) \equiv \bar{L}(t) - G(t)$ where

$$\bar{L}(t)(\cdot) \equiv -\sum_{i,j=1}^{d} D_i(a_{ij}D_j(\cdot)) + (a_0 - \frac{1}{2}\sum_{i=1}^{d} D_i a_{0i})(\cdot),$$

$$G(t)(\cdot) \equiv \sum_{i=1}^{d} a_{0i}D_i(\cdot) + \frac{1}{2}(\sum_{i=1}^{d} D_i a_{0i})(\cdot) \quad .$$

We shall assume that $\bar{a}_0 \equiv (a_0 - \frac{1}{2}\sum_{i=1}^{d} D_i a_{0i}) \geq 0$ on $\bar{\Omega} \times [0,\tau]$ if we are working with the Dirichlet boundary conditions or that $\bar{a}_0 > 0$ on $\bar{\Omega} \times [0,\tau]$ if we are workint with the Neumann boundary conditions. Thus $D(t)$ and $D^*(t)$ are (strongly) coercive forms over H^1_o if we have the Dirichlet boundary conditions or over H^1 if we have the Neumann boundary conditions.

Since we wish to regard L, L^* and $\bar{L}$ as unbounded operators on $L^2(\Omega)$, we must discuss their (common) domain of definition D_L. If the Dirichlet conditions are under consideration, we let $D_L \equiv H^2 \cap H^1_o$. If we are considering the Neumann problem, we let $D_L \equiv H^2 \cap \{w \in H^2 : \sum_{i,j=1}^{d} \tilde{a}_{ij}n_iD_jw\big|_{\partial\Omega} = 0\}$. Then L, L^* and $\bar{L}$ are indeed closed, unbounded operators on $L^2(\Omega)$ with a common domain D_L, L^* is the $L^2(\Omega)$-adjoint of L and $\bar{L}$ is selfadjoint. We will give D_L the $\|\cdot\|_2$-norm and, for convenience, we will give $G(t)$ and $G^*(t) \equiv -G(t)$ the domain D_L.

We will now identify a space intermediate to $L^2(\Omega)$ and D_L that will be useful later. Let $0 \leq t \leq \tau$. Since $\bar{L}(t)$ is selfadjoint and positive definite, we can use spectral theory to define $\bar{L}^{\frac{1}{2}}(t)$ on its domain $D(\bar{L}^{\frac{1}{2}}(t)) \subset L^2(\Omega)$ and we can give the latter the norm $\|\bar{L}^{\frac{1}{2}}(t)(\cdot)\|$. If $g \in D_L$, then

$$\|\bar{L}^{\frac{1}{2}}g\|^2 = (\bar{L}g,g) = \frac{1}{2}(D + D^*)(g,g) \simeq \|g\|_1^2$$

where we have used $\simeq$ to denote a norm equivalence. Thus it can be seen that

$D(\bar{L}^{\frac{1}{2}}(t))$ is the $\|\cdot\|_1$-norm closure of D_L and the $\|\bar{L}^{\frac{1}{2}}(t)(\cdot)\|$-norm is equivalent to the $\|\cdot\|_1$-norm. We will write H^1_* to denote this space, which is H^1_o if the Dirichlet boundary conditions are under consideration or H^1 if the Neumann conditions are being used and we will give it the $\|\cdot\|_1$-norm.

We let $T(t)$ denote the solution operator for the elliptic boundary value problem associated with $L(t)$, for $0 \le t \le \tau$. Thus $T(t)$ is an operator from right hand sides $g \in H^\ell$, for any $\ell \ge 0$, to solutions in $H^{\ell+2} \cap D_L$ that satisfy $L(t)(T(t)g) = g$. We define $T^*(t)$ and $\bar{T}(t)$ analogously and note that $T^*(t)$ is indeed the $L^2(\Omega)$-adjoint of $T(t)$ and that $\bar{T}(t)$ is selfadjoint. (We will continue to use the symbol * to denote adjoints taken with respect to the $L^2(\Omega)$-inner product).

Let $j \ge 0$ and let $L^{(j)}(t) \equiv \left(\frac{d}{dt}\right)^j L(t)$ denote the operator obtained by differentiating the coefficients of $L(t)$ with respect to time. We give this family of operators the domain D_L and we define $L^{*(j)}(t)$ and $\bar{L}^{(j)}(t)$ similarly. If we regard $T(t) : H^\ell \to H^{\ell+2} \cap D_L$ *for some* $\ell \ge 0$, *we can verify that* $T^{(1)} = (\frac{d}{dt})T$ exists (in the operator norm) and is, in fact, $T^{(1)} = -TL^{(1)}T$. We can continue differentiating and show that $T^{(j)}(t)$ exists for each $j \ge 0$. Similar statements hold for $T^*(t)$ and $\bar{T}(t)$.

We can now use the work of Sobolevskii [10] to study the solution of (1.1). The solution u of (1.1) can be described via a family of fundamental solution operators $U(t,s) \in B(L^2(\Omega))$, defined for $0 \le s \le t \le \tau$. In fact, the operators $U(t,s)$ are strongly continuous in $L^2(\Omega)$ for $0 \le s \le t \le \tau$, are continuously differentiable in each variable in $B(L^2(\Omega))$ and have range in D_L for $0 \le s < t \le \tau$ and are characterized by the following equations:

$$\frac{\partial}{\partial t} U(t,s) + L(t)U(t,s) = 0 \quad \text{and} \quad U(s,s) = I \quad \text{for} \quad 0 \le s < t \le \tau \quad . \tag{2.1}$$

The unique solution of (1.1) is given by $u(t) = U(t,0)v$, for $0 \le t \le \tau$. We note that $U(t,\xi)U(\xi,s) = U(t,s)$ for $0 \le s \le \xi \le t \le \tau$ and that $\frac{\partial}{\partial s} U(t,s) = U(t,s)L(s)$ on D_L for $0 \le s < t \le \tau$.

We can use such fundamental solution operators to describe the solution of a more general problem than (1.1). The following equations:

(2.2)(i) $\qquad w_t + L(t)w = f(t) \quad \text{for} \quad 0 < t \leq \tau \quad \text{and} \quad w(0) = w_0 \quad ,$

where $w_0 \in L^2(\Omega)$ and $f \in C^\epsilon([0,\tau],L^2(\Omega))$ are known and $\epsilon > 0$, have a unique solution

(2.2)(ii) $\qquad w \in C([0,\tau],L^2(\Omega)) \cap C^1((0,\tau],L^2(\Omega)) \cap C((0,\tau],D_L)$

given by

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)f(s)ds \quad \text{for} \quad 0 \leq t \leq \tau \quad . \tag{2.3}$$

If $w_0 \in D_L$, then $w \in C([0,\tau],D_L)$ and if $w_0 = f(0) = 0$, $w \in C^{\epsilon_1}([0,\tau],D_L)$ for some $\epsilon_1 > 0$. Moreover, if $\ell \geq 0$ and $f \in C^{\ell+1}([0,\tau],L^2(\Omega))$, then $w \in C^{\ell+1}([0,\tau],L^2(\Omega))$ and

$$\left\|\left(\frac{d}{dt}\right)^{\ell+1} w(t)\right\| \leq C(\ell)t^{-\ell-1}\left(\|w_0\| + \sup_{\substack{0\leq s\leq t \\ 0\leq j\leq \ell+1}} \left\|\left(\frac{d}{ds}\right)^j f(s)\right\|\right) \quad \text{for} \quad 0 < t \leq \tau \quad , \tag{2.4}$$

for some constant $C(\ell)$. (We will often write $w^{(j)}(t) \equiv \left(\frac{d}{dt}\right)^j w(t)$, $u^{(j)}(t) \equiv \left(\frac{d}{dt}\right)^j u(t)$ and $U^{(j)}(t,s) \equiv \left(\frac{\partial}{\partial t}\right)^j U(t,s)$ for $j \geq 0$, in the future).

We also note that other results from [10] show that if $\bar{w}$ has the continuity properties described in (2.2)(ii) and $\bar{w}_t + \bar{L}\bar{w} = 0$ for $0 < t \leq \tau$ where $\bar{w}(0) \in H^1_\star = D(\bar{L}^{\frac{1}{2}}(0))$, then $\bar{w} \in C^\epsilon([0,\tau],L^2(\Omega))$ and $(t\bar{w}(t)) \in C^\epsilon([0,\tau],D_L)$, for some $\epsilon > 0$. These results allow us to derive the following:

<u>Proposition (2.1)</u>: If $w_0 \in D_L$, $f \in C^\epsilon([0,\tau],L^2(\Omega))$ for some $\epsilon > 0$ and $L(0)w_0 - f(0) \in H^1_\star$, then the solution w of (2.2) is in $C^{\epsilon_1}([0,\tau],D_L)$ for some $\epsilon_1 > 0$.

<u>Proof</u>: Let $y_t + \bar{L}(t)y = 0$ for $0 < t \leq \tau$ and $y(0) = L(0)w_0 - f(0) \in H^1_\star$. Then if $z \equiv w(t) - w_0 + ty(t)$ for $0 \leq t \leq \tau$, there is an $\epsilon_2 > 0$ so that

$$z_t + Lz = (f - Lw_0 + y + (L - \bar{L})(ty)) \equiv g \in C^{\epsilon_2}([0,\tau],L^2(\Omega))$$

and $z(0) = g(0) = 0$. Thus $z \in C^{\epsilon_3}([0,\tau],D_L)$ for some $\epsilon_3 > 0$ and the result follows.

We will soon want to examine further conditions under which the solution of (2.2) is smooth at $t = 0$ and we will also want to estimate the size of this solution in terms of data. We shall study another result about solutions of (2.2) to carry out this analysis.

Proposition (2.2): If $f \in C^1([0,\tau],L^2(\Omega))$ is such that there is a (unique) solution $w \in C^1([0,\tau],L^2(\Omega)) \cap C([0,\tau],D_L)$ of (2.2) satisfying $w(0) = w_0 = 0$, then for each $\epsilon > 0$, there is a constant $C = C(\epsilon) > 0$ so that

$$\|w(t)\| \le C \sup_{0\le s\le t} \|Tf(s)\| + \epsilon \sup_{0\le s\le t} s\|(Tf)_s(s)\| \quad \text{for } 0 \le t \le \tau \quad , \tag{2.5}$$

$$\|w(t)\|_1 \le C \sup_{0\le s\le t} \|Tf(s)\|_1 + \epsilon \sup_{0\le s\le t} s\|Tf_s(s)\|_1 \quad \text{for } 0 \le t \le \tau \quad . \tag{2.6}$$

Proof: We see that $Tw_t + w = Tf$ and $w(0) = 0$. We will analyze this equation using energy techniques that were used in a similar argument in [3].

We first see that if $0 < t \le \tau$, then

$$t\|w(t)\|^2 = \int_0^t \|w\|^2 ds + \int_0^t s \frac{d}{ds}\|w(s)\|^2 ds \quad . \tag{2.7}$$

The equation, (2.2), shows that

$$\frac{1}{2}\frac{d}{ds}\|w\|^2 \le \frac{1}{2}\frac{d}{ds}\|w\|^2 + (Tw_s,w_s) = (Tf,w_s) \tag{2.8}$$

$$= (Tf,w)_s - ((Tf)_s,w) \quad \text{for } 0 < s \le \tau \quad .$$

Thus, if we integrate (2.8), we find that

$$\int_0^t s\frac{d}{ds}\|w\|^2 ds \le \frac{t}{2}\|w\|^2 + Ct\|Tf\|^2 + C\int_0^t \|Tf\|^2 ds + C\int_0^t \|w\|^2 ds \tag{2.9}$$

$$+ \frac{\epsilon}{2} t \sup_{0\le s\le t} s^2\|(Tf)_s\|^2 \quad .$$

Since $\bar{T}w_t + w = \bar{T}f - \bar{T}Gw$, we see that

$$(2.10)\qquad (\bar{T}w_s, w) + \|w\|^2 = \frac{1}{2}(\bar{T}w,w)_s - \frac{1}{2}(\bar{T}^{(1)}w,w) + \|w\|^2$$

$$= (\bar{T}f,w) - (\bar{T}Gw,w) \quad \text{for} \quad 0 < s \leq \tau \quad .$$

We now analyze the individual terms in (2.10). We have the following estimates, where $\epsilon_1 > 0$:

$$(2.11)\qquad (\bar{T}^{(1)}w,w) = -(\bar{L}^{(1)}\bar{T}w,\bar{T}w) \leq C\|\bar{T}w\|_1^2$$

$$\leq C(\bar{L}\bar{T}w,\bar{T}w) = C(\bar{T}w,w) \quad ,$$

$$(2.12)\qquad -(\bar{T}Gw,w) \leq C(\bar{T}w,w) + \epsilon_1(\bar{T}Gw,Gw) \quad ,$$

$$(2.13)\qquad (\bar{T}Gw,Gw) = -(G\ \bar{T}Gw,w) \leq C\|\bar{T}Gw\|_1 \|w\|$$

$$\leq C(\bar{L}\bar{T}Gw,\bar{T}Gw)^{\frac{1}{2}} \|w\| = C(\bar{T}Gw,Gw)^{\frac{1}{2}} \|w\|$$

(that is, $(\bar{T}Gw,Gw) \leq C\|w\|^2$) and

$$(2.14)\qquad \|\bar{T}f\| = \|\bar{T}(\bar{L} + G)Tf\| \leq C\|Tf\| \quad .$$

Thus, returning to (2.10), we can now see that a suitable choice for $\epsilon_1 > 0$ leads to the following:

$$(2.15)\qquad (\bar{T}w,w)_s + \|w\|^2 \leq C(\bar{T}w,w) + C\|Tf\|^2 \quad \text{for} \quad 0 < s \leq \tau \quad .$$

This implies that

$$(2.16)\qquad \int_0^t \|w\|^2 ds \leq C \int_0^t \|Tf\|^2 ds \quad .$$

We can now obtain (2.5) from (2.7), (2.9) and (2.16).

We now turn to (2.6). We first see that if $0 < t \leq \tau$, then

$$Ct\ \|w\|_1^2 \leq t(\bar{L}w,w)$$

$$= \int_0^t (Lw,w)ds + \int_0^t s(\bar{L}^{(1)}w,w)ds + 2\int_0^t s(\bar{L}w,w_s)ds$$

$$\leq C\int_0^t (\bar{L}w,w)ds + 2\int_0^t s(\bar{L}w,w_s)ds \quad .$$

Since $w_s + \bar{L}w = f - Gw$, we see that

$$\|w_s\|^2 + (\bar{L}w,w_s) = (Tf,L^*w)_s - (Tf_s,L^*w) - (Gw,w_s) \quad \text{for} \quad 0 < s \le \tau \quad .$$

Thus integration gives us the following, where $\epsilon_1 > 0$:

$$\int_0^t s\|w_s\|^2 ds + 2\int_0^t s(\bar{L}w,w_s)ds \le \frac{1}{2}t(\bar{L}w,w) + Ct\|Tf\|_1^2$$

$$+ C\int_0^t \|Tf\|_1^2 ds + C\int_0^t (\bar{L}w,w)ds + \epsilon_1\int_0^t s^2\|Tf_s\|_1^2 ds \quad .$$

Returning to (2.2), we find that

$$\frac{1}{2}\frac{d}{ds}\|w\|^2 + (Lw,w) = (Tf,L^*w) \le C\|Tf\|_1^2 + \frac{1}{2}(Lw,w) \quad \text{for} \quad 0 < s \le \tau \quad ,$$

which gives us the following estimate:

$$\int_0^t (\bar{L}w,w)ds = \int_0^t (Lw,w)ds \le C\int_0^t \|Tf\|_1^2 ds \quad .$$

We can now obtain (2.6), which completes the proof.

We now study results that hold for the solution u of (1.1).

<u>Proposition (2.3)</u>: We have the following for $0 < t \le \tau$ and $m \ge 0$:

$$u^{(m+1)} + Lu^{(m)} = -\sum_{\ell=0}^{m-1}\binom{m}{\ell}L^{(m-\ell)}u^{(\ell)} \quad , \tag{2.17}$$

$$u^{(m)} = -T\sum_{\ell=0}^{m}\binom{m}{\ell}(LT^{(m-\ell)})u^{(\ell+1)} \quad . \tag{2.18}$$

<u>Proof</u>: Since $u \in C^{m+1}((0,\tau],L^2(\Omega))$, we can obtain (2.18) by successive differentiation of the equation $Tu_t + u = 0$. This shows that $u \in C^m((0,\tau],D_L)$. Successive differentiation of (1.1) then gives (2.17) and completes the proof.

Proposition (2.3) will be used extensively, in conjunction with elliptic regularity results, to translate information about derivatives in time of u to information about spacial norms of u. For instance, an inductive argument shows that

$$\|u(t)\|_{\begin{Bmatrix}2m\\2m+1\end{Bmatrix}} \le C\sum_{\ell=0}^{m}\|u^{(\ell)}(t)\|_{\begin{Bmatrix}0\\1\end{Bmatrix}} \quad \text{for} \quad m \ge 0 \quad \text{and} \quad 0 < t \le \tau \quad .$$

We now derive a series of estimates that relate norms of the solution u at one time to norms of equal or less weight at an earlier time, giving up a constant that contains a pole in the time increment. These estimates will prove important in later applications.

Proposition (2.4): If $0 \leq \ell \leq m$ and $0 < s < t \leq \tau$, we have that

(2.19)
$$(t-s)^{\ell}\|u^{(m)}(t)\| \quad , \quad (t-s)^{\ell}\|u(t)\|_{2m}$$
$$\leq C\sum_{j=0}^{m-\ell}\|u^{(j)}(s)\| \leq C\|u(s)\|_{2(m-\ell)} \quad ,$$

(2.20)
$$(t-s)^{\ell}\|u^{(m)}(t)\|_1 \quad , \quad (t-s)^{\ell}\|u(t)\|_{2m+1}$$
$$\leq C\sum_{j=0}^{m-\ell}\|u^{(j)}(s)\|_1 \leq C\|u(s)\|_{2(m-\ell)+1} \quad ,$$

(2.21)
$$(t-s)^{m+\frac{1}{2}}\|u^{(m)}(t)\|_1 \leq C\|u(s)\| \quad .$$

Proof: We will prove (2.19) and (2.20) first and it will suffice to prove these results with $m \geq 0$ and $\ell = 0$ or 1, for just the time derivatives. If $\ell > 1$, the interval $[s,t]$ can be split into ℓ equal pieces and the results for $\ell = 1$ used ℓ times. Equations (2.17) and (2.18) and elliptic regularity can be used to obtain the results for the spacial derivatives. We note that we are avoiding the case $s = 0$, so we have sufficient smoothness.

If $m \geq 0$ and $0 < t \leq \tau$, (2.17) shows that

$$\frac{1}{2}\frac{d}{dt}\|u^{(m)}(t)\|^2 + (Lu^{(m)},u^{(m)}) \leq C\sum_{j=0}^{m-1}\|u^{(j)}\|_2\|u^{(m)}\|$$
$$\leq C\|u^{(m)}(t)\|^2 + C\sum_{j=0}^{m-1}\|u^{(j)}(t)\|^2 \quad .$$

We can now obtain (2.19) for $m \geq 0$ and $\ell = 0$ by an induction argument. (Note that $\|u(t)\| \leq \|u(s)\|$ for $0 \leq s \leq t \leq \tau$).

We also have the following for $0 < t \le \tau$:

$$\|u^{(m+1)}(t)\|^2 + \frac{1}{2}(\bar{L}u^{(m)},u^{(m)})_t$$

$$\le C\sum_{j=0}^{m-1}\|u^{(j)}\|_2\,\|u^{(m+1)}\| + C\|u^{(m)}\|_1^2 + \frac{1}{4}\|u^{(m+1)}\|^2$$

$$\le \frac{1}{2}\|u^{(m+1)}\|^2 + C(\bar{L}u^{(m)},u^{(m)}) + C\sum_{j=0}^{m-1}\|u^{(j)}\|_1^2 \quad .$$

Thus, we can again use an inductive argument to obtain a result, in this case (2.20) with $m \ge 0$ and $\ell = 0$.

Now set $w(t) \equiv tu^{(m)}(t + s)$ where $m \ge 1$, $s > 0$ and $0 \le t \le \tau - s$. Then

$$w_t + Lw = -t\sum_{j=0}^{m-1}\binom{m}{j}L^{(m-j)}u^{(j)} + u^{(m)}$$

$$= -t\sum_{j=0}^{m-1}\binom{m}{j}L^{(m-j)}u^{(j)} - \sum_{j=0}^{m-1}\binom{m-1}{j}L^{(m-1-j)}u^{(j)} \equiv f \quad \text{for} \quad 0 < t \le \tau - s$$

and $w(0) = 0$. Using Proposition (2.2), we see that if $0 \le \xi \le \tau - s$, then

$$\|w(\xi)\| = \xi\|u^{(m)}(\xi + s)\| \le C\sum_{j=0}^{m-1}\|u^{(j)}(s)\| + \frac{1}{2}\sup_{0\le\zeta\le\xi}\zeta\|u^{(m)}(\zeta + s)\| \quad .$$

This gives (2.19) with $\ell = 1$ and $m \ge 1$, after a change of variables is made. We now derive (2.20) the same way, using the $\|\cdot\|_1$-norm results of Proposition (2.1).

To prove (2.21), we observe that (2.17) and (2.19) show that

$$(t - s)^{m+1}\|u^{(m)}(t)\|_2 \le C(t - s)^{m+1}\|L(t)u^{(m)}(t)\| \le C\|u(s)\| \quad ,$$

and that $(t - s)^m\|u^{(m)}(t)\| \le C\|u(s)\|$, for $0 < s \le t \le \tau$. Thus we can use an interpolation argument to see that

$$\|u^{(m)}(t)\|_1 \le C(\bar{L}(t)u^{(m)}(t),u^{(m)}(t))^{\frac{1}{2}} \le C\|u^{(m)}(t)\|^{\frac{1}{2}}\,\|\bar{L}(t)u^{(m)}(t)\|^{\frac{1}{2}}$$

$$\le C\|u^{(m)}(t)\|^{\frac{1}{2}}\,\|u^{(m)}(t)\|_2^{\frac{1}{2}} \le C(t - s)^{-m-\frac{1}{2}}\|u(s)\| \quad .$$

This completes the proof.

We will now study when u is smooth at $t = 0$. The initial data v must satisfy boundary compatibility conditions for this to occur. We will use certain (unbounded) "*time differentiation*" operators $A^{(m)}(t)$, defined for $m \geq 0$ and $0 \leq t \leq \tau$, to study these conditions. These operators will satisfy the equation $u^{(m)}(t) = A^{(m)}(t)u(t)$ for $0 < t \leq \tau$ and their form will be motivated by (2.17). Fix $0 \leq t \leq \tau$, let $A^{(0)}(t) \equiv I$ on $D(A^{(0)}(t)) \equiv L^2(\Omega)$ and let

$$A^{(m+1)}(t) \equiv - \sum_{j=0}^{m} \binom{m}{j} L^{(m-j)}(t) A^{(j)}(t) \quad ,$$

provided that $\{A^{(j)}(t)\}_{j=0}^{m}$ are given, be defined on the domain

$$D(A^{(m+1)}(t)) \equiv \{w : w \in D(A^{(m)}(t)) \text{ and } A^{(m)}(t)w \in D_L\} \quad .$$

We note that $A^{(1)}(t) = -L(t)$, $A^{(2)}(t) = (L^2 - L^{(1)})(t)$ and that elliptic regularity shows that $D(A^{(m)}(t)) \subset H^{2m}$ for $m \geq 0$. Moreover,

$$C_c^{\infty}(\Omega) \equiv \{f \in C^{\infty}(\Omega) : \text{supp } f \subset \Omega\} \subset D(A^{(m)}(t)) \quad \text{for } m \geq 0 \quad .$$

We will now use these time derivative operators to characterize when the solution u of (1.1) is smooth at time zero.

<u>Proposition (2.5)</u>: (1) If $v \in D(A^{(m)}(0))$ and $A^{(m)}(0)v \in H_*^1$ for some $m \geq 0$, then

$$u \in C^m([0,\tau],H_*^1) \cap C([0,\tau],H^{2m+1}) \tag{2.22}$$

and $u \in C^{m-1}([0,\tau],D_L)$ if $m \geq 1$.

(2) If $v \in D(A^{(m+1)}(0))$ for some $m \geq 0$ then

$$u \in C^{m+1}([0,\tau],L^2(\Omega)) \cap C^m([0,\tau],D_L) \cap C([0,\tau],H^{2m+2}) \quad . \tag{2.23}$$

We note that this Proposition indicates when the restriction "$s > 0$" can be removed in Proposition (2.4), via the taking of limits, when v satisfies the correct compatibility conditions.

In the future, if we say that $v \in D(A^{(m+\frac{1}{2})}(t))$ for some $m \geq 0$ and $0 \leq t \leq \tau$, we will mean that $v \in D(A^{(m)}(t))$ and $A^{(m)}(t)v \in H_*^1$.

Proof: Our previous discussions indicate when equations of the form (2.2) have limits in D_L as $t \to 0$; that is, when $w_0 \in D_L$ and $f \in C^{\epsilon}([0,\tau],L^2(\Omega))$ for some $\epsilon > 0$. Now we will study the H^1_* case. If $z \in D_L$, (2.20) shows that $\|U(t,0)z\|_1 \leq C\|z\|_1$ for $0 < t \leq \tau$. Thus, by density, $U(t,0) : H^1_* \to H^1_*$ is uniformly bounded for $0 < t \leq \tau$. Since $U(t,0)z \to z$ in H^1_* as $t \to 0$ if $z \in D_L$, $U(t,0)z \to z$ in H^1_* as $t \to 0$ if $z \in H^1_*$. Thus if w satisfies (2.2) where $w_0 \in H^1_*$ and $f \in C^{\epsilon}([0,\tau],L^2(\Omega))$ for some $\epsilon > 0$, $w(t) \to w_0$ in H^1_* as $t \to 0$.

Equation (2.17) now suggests an induction argument which would use the above H^1_* and D_L convergence results, which would fit with our definition of $D(A^{(m)}(0))$ and which would complete the proof. However it would be necessary to know when the right hand side of (2.17) is in $C^{\epsilon}([0,\tau],L^2(\Omega))$ for some $\epsilon > 0$, since the induction hypothesis would only indicate that the right hand side is continuous. The required result is discussed in Proposition (2.1), so the induction argument can be carried out.

Thus, we can prove the results on the time derivatives of u. Equations (2.17) and (2.18) and elliptic regularity then complete the proof.

We now wish to identify the $L^2(\Omega)$-adjoint of U, the fundamental solution operator. This identification will prove useful later in some bootstrapping arguments (as in Helfrich [5]). We state it here and supply a proof.

Let $\tilde{L}(t) = L^*(\tau - t)$ define another family of differential operators. We note that these operators have all the properties required of the family $\{L(t)\}$. Thus the equations

$$\dot{\tilde{U}}(t,s) + \tilde{L}(t)\tilde{U}(t,s) = 0 \quad \text{and} \quad \tilde{U}(s,s) = I \quad , \quad \text{for} \quad 0 \leq s \leq t \leq \tau \quad ,$$

define a fundamental solution operator $\tilde{U}$ to which parabolic regularity applies. We have the following result.

Proposition (2.6): $U^*(t,s) = \tilde{U}(\tau - s, \tau - t)$ for $0 \leq s \leq t \leq \tau$.

Proof: Let $\hat{U}(t,s) = \tilde{U}^*(\tau - s, \tau - t)$ for $0 \leq s \leq t \leq \tau$ and note that $\frac{\partial}{\partial t}\hat{U}(t,s)$ exists in $B(L^2(\Omega))$ if $t \neq s$. Thus if $f \in L^2(\Omega)$, $g \in D_L$ and $s < t$, then

$$(\hat{U}_t f,g) = (f,\dot{U}(\tau - s,\tau - t)_t g) = (\hat{U}f,L^*(t)g) = (L\hat{U}f,g) \quad ;$$

the last step is valid since the previous steps showed that $\hat{U}f \in D_L$. Thus, $\hat{U}$ satisfies (2.1). Moreover, if $0 \leq s < t \leq \tau$ and f and $g \in D_L$, an estimate from [10] shows that $|(\hat{U}(t,s)f - f,g)| \leq C|t - s| \, \|f\|_2 \, \|g\|$. Thus $\hat{U}$ is strongly continuous on D_L for $0 \leq s \leq t \leq \tau$. Since $\|\hat{U}\| = \|U\| \leq 1$ for $0 \leq s \leq t \leq \tau$, $\hat{U}$ is strongly continuous on $L^2(\Omega)$. This completes the proof.

We conclude this Section with results of a technical nature concerning the fine structure of the $A^{(m)}(t)$-operators. Let $0 \leq t \leq \tau$ and $K \geq 0$. Define $L_+(t) \equiv L(t) + K$ on D_L and let $T_+(t) \in B(L^2(\Omega))$ be the associated solution operator; that is, $L_+T_+ = I$ on $L^2(\Omega)$. We define $L_+^{(j)}(t)$ and $T_+^{(j)}(t)$ as before, for $j \geq 0$. Let $A_+^{(0)}(t) = I$ and, for $m \geq 0$, define the following (inductively) on $D(A^{(m+1)}(t))$:

$$(2.24) \quad A_+^{(m+1)}(t) \equiv - \sum_{\ell=0}^{m} \binom{m}{\ell} L_+^{(m-\ell)}(t) A_+^{(\ell)}(t) = - \Big(\sum_{\ell=0}^{m} \binom{m}{\ell} L^{(m-\ell)} A_+^{(\ell)} + KA_+^{(m)} \Big) \quad .$$

Note that $A_+^{(m)}(t)(e^{-Kt}u(t)) = \left(\frac{d}{dt}\right)^m (e^{-Kt}u(t))$ if $m \geq 0$ and $0 < t \leq \tau$ and

$$(2.25) \qquad \sum_{\ell=0}^{m} \binom{m}{\ell} K^{m-\ell} A_+^{(\ell)} u = \sum_{\ell=0}^{m} \binom{m}{\ell} (K^{m-\ell} e^{Kt})(e^{-Kt}u)^{(\ell)} = A^{(m)} u$$

if $m \geq 0$ and $0 < t \leq \tau$. Letting v range over $D(A^{(m)}(0))$ shows that we can take $t = 0$ in (2.25). Suitable translations of the origin $t = 0$ then show that

$$(2.26) \qquad A^{(m)}(t) = \sum_{\ell=0}^{m} \binom{m}{\ell} K^{m-\ell} A_+^{(\ell)}(t) \quad \text{on } D(A^{(m)}(t)), \text{ for } 0 \leq t \leq \tau \quad .$$

Now let $E_+^{(0)}(t) \equiv I$, $E_+^{(1)}(t) \equiv L_+(t)$ and, for each $m \geq 1$, define $E_+^{(m+1)}(t)$ as follows on D_L, whenever each $E_+^{-(j)}(t) \equiv (E_+^{(j)}(t))^{-1}$ exists in $B(L^2(\Omega))$, for $0 \leq j \leq m$:

$$(2.27) \qquad E_+^{(m+1)}(t) \equiv L_+(I + mT_+^{(1)} + \sum_{\ell=0}^{m-2} (-1)^{m+1-\ell} \binom{m}{\ell} T_+^{(m-\ell)} E_+^{-(\ell+2)} \cdots E_+^{-(m)}) \quad .$$

We have the following:

Proposition (2.7): For each $m \geq 1$, there is a $\hat{K} = \hat{K}(m) \geq 0$ so that the following hold for $1 \leq \ell \leq m$, $0 \leq t \leq \tau$ and $K \geq \hat{K}$:

(2.28) $\quad E_+^{(\ell)} : H^{i+2} \cap D_L \to H^i$ exists and is bounded for $i \geq 0$,

(2.29) $\quad E_+^{-(\ell)} : H^i \to H^{i+2} \cap D_L$ exists and is bounded for $i \geq 0$,

(2.30) $\quad A_+^{(m)} = (-1)^m E_+^{(m)} \ldots E_+^{(1)}$ on $D(A^{(m)})$.

Proof: Let $0 \leq t \leq \tau$, $K \geq 0$ and $f \in L^2(\Omega)$. Since $L_+T_+f = LT_+f + KT_+f$,

$$(C + K) \|T_+f\|^2 \leq (LT_+f,T_+f) + K\|T_+f\|^2 = (f,T_+f) \leq \|f\| \, \|T_+f\| \quad .$$

Thus $(C + K) \|T_+f\| \leq C\|f\|$. Also $\|T_+f\|_2 \leq C\|LT_+f\| \leq C\|f\|$. Moreover, $\|L_+T_+f\| = \|f\|$ and if $\ell \geq 1$,

$$\|L_+T_+^{(\ell)}f\| \leq \sum_{j=0}^{\ell-1} \binom{\ell}{j} \|L^{(\ell-j)}T_+^{(j)}f\| \quad ,$$

so that by induction, $\|L_+T_+^{(\ell)}\| \leq C$ for $\ell \geq 0$. Thus

$$(C + K) \|T_+^{(\ell)}f\| = (C + K) \|T_+L_+T_+^{(\ell)}f\| \leq C\|f\| \quad \text{for } \ell \geq 0 \quad .$$

We know that $E_+^{(1)}$ is well defined, (2.28) and (2.29) are satisfied, $A_+^{(1)} = -E_+^{(1)}$ and $(C + K) \|E_+^{-(1)}\| \leq C$. We will now assume, for some $1 \leq \ell \leq m - 1$, that $E_+^{(1)},\ldots,E_+^{(\ell)}$ are well defined, (2.28) and (2.29) are satisfied, $A_+^{(j)} = -E_+^{(j)}A_+^{(j-1)}$ for $1 \leq j \leq \ell$ and $(C + K) \|E_+^{-(j)}\| \leq C$ for $1 \leq j \leq \ell$. Equation (2.27) then defines $E_+^{(\ell+1)}$ and the inductive assumptions show that (2.28) follows for $E_+^{(\ell+1)}$.

Note that $E_+^{(\ell+1)} = L_+ + \ell L_+T_+^{(1)} + R_+^{(\ell+1)} = L_+ + B_+^{(\ell+1)}$ where $(C + K) \|R_+^{(\ell+1)}\| \leq C$ and $\|L_+T_+^{(1)}\| \leq C$. Thus we can choose $K \geq 0$ so that $2\|B_+^{(\ell+1)}\| \leq K$. Then for any $g \in L^2(\Omega)$, there is an unique $w \in H_*^1$ that satisfies:

(2.31) $$D(w,\varphi) + K(w,\varphi) + (B_+w,\varphi) = (g,\varphi) \quad \text{for all } \varphi \in H_*^1 \quad ,$$

since the form on the left hand side of (2.31) is coercive over H^1_*. But since $D(w,\varphi) = (g - Kw - B_+ w, \varphi)$ for all $\varphi \in H^1_*$, elliptic regularity implies that $w \in D_L$. Moreover,

$$(C + K/2)\,\|w\|^2 \leq D(w,w) + K(w,w) + (B_+ w,w) = (g,w) \quad ,$$

so $(C + K)\,\|w\| \leq C\|g\|$. Thus $E_+^{(\ell+1)} : D_L \to L^2(\Omega)$ is invertible and $(C + K)\,\|E_+^{-(\ell+1)}\| \leq C$. It is now easily checked that (2.29) holds for $E_+^{-(\ell+1)}$.

We will now study (2.30). On $D(A^{(\ell+1)})$, we have that

$$E_+^{(\ell+1)} A_+^{(\ell)} = L_+ (A_+^{(\ell)} + \ell T_+^{(1)} A_+^{(\ell)}$$

$$+ \sum_{j=0}^{\ell-2} (-1)^{\ell-j-1} \binom{\ell}{j} T_+^{(\ell-j)} E_+^{-(j+2)} \ldots E_+^{-(\ell)} A_+^{(\ell)})$$

$$= L_+ (A_+^{(\ell)} + \sum_{j=0}^{\ell-1} \binom{\ell}{j} T_+^{(\ell-j)} A_+^{(j+1)}) \quad .$$

But since

$$- \sum_{j=0}^{\ell-1} \binom{\ell}{j} L_+ T_+^{(\ell-j)} A_+^{(j+1)} = \sum_{j=0}^{\ell-1} \binom{\ell}{j} L_+ T_+^{(\ell-j)} \sum_{i=0}^{j} \binom{j}{i} L_+^{(j-i)} A_+^{(i)}$$

$$= \sum_{i=0}^{\ell-1} \sum_{j=i}^{\ell-1} \binom{\ell}{j} \binom{j}{i} L_+ T_+^{(\ell-j)} L_+^{(j-i)} A_+^{(i)} \qquad (j \to j + i)$$

$$= \sum_{i=0}^{\ell-1} \binom{\ell}{i} L_+ (\sum_{j=0}^{\ell-1-i} \binom{\ell-i}{j} T_+^{(\ell-j-i)} L_+^{(j)}) A_+^{(i)}$$

$$= \sum_{i=0}^{\ell-1} \binom{\ell}{i} L_+ ((T_+ L_+)^{(\ell-i)} - T_+ L_+^{(\ell-i)}) A_+^{(i)}$$

$$= - \sum_{i=0}^{\ell-1} \binom{\ell}{i} L_+^{(\ell-i)} A_+^{(i)} \quad ,$$

we see that $A_+^{(\ell+1)} = -E_+^{(\ell+1)} A_+^{(\ell)}$.

The proof can now be completed by induction.

Under the conditions of this last result, we see that if $m \geq 1$, the operators $A_{+}^{(m)}$ can be factored into operators that are each a bounded perturbation of L and which are in fact L if we are dealing with time independent coefficients. Moreover, if we regard $A_{+}^{(m)}$ as an operator from $D(A^{(m)})$ (to which we give the $\|\cdot\|_{2m}$-norm) to $L^2(\Omega)$ if $m \geq 0$, it has a bounded inverse which we will denote by $A_{+}^{-(m)}$.

III. The Semidiscrete Approximation

In the last Section, we discussed some properties of the parabolic equation we intend to study. Now we are going to examine a method of constructing an approximation to its solution, given a method of approximating solutions of the associated elliptic problem. We begin by stating the kind of properties we expect such elliptic approximation methods to have, although we defer verification of these properties for several standard methods to a later Section. We then define the semidiscrete approximation to the solution of the parabolic problem and begin our study of it.

We assume that we are given a finite dimensional subspace $S_h \subset L^2(\Omega)$ (that depends on a small parameter $0 < h < 1$) and a family of approximate elliptic differential equation solvers $\{T_h(t)\}_{0 \leq t \leq \tau}$ that are (at least) bounded operators on $L^2(\Omega)$. We will give S_h the $L^2(\Omega)$-inner product. We will further require that the following hold for $0 \leq t \leq \tau$:

(3.1)(i) $T_h(t)$, $T_h^*(t)$: $L^2(\Omega) \to S_h$,

(3.1)(ii) $(T_h f, f) \geq 0$ for $f \in L^2(\Omega)$ with strict inequality if $0 \neq f \in S_h$.

Since $L_h \equiv (T_h|_{S_h})^{-1} : S_h \to S_h$ exists, we can ask that

(3.1)(iii) $L_h^{(\ell)}(t) \equiv \left(\frac{d}{dt}\right)^{\ell} L_h(t) : S_h \to S_h$ exists for $\ell \geq 0$.

Finally, setting $G_h(t) \equiv \frac{1}{2}(L_h(t) - L_h^*(t))$, we will require that

(3.1)(iv) $\|G_h(t)\varphi\|^2, \|\varphi\|^2 \leq C_+(L_h(t)\varphi,\varphi)$, for all $0 \leq t \leq \tau$ and $\varphi \in S_h$,

where C_+ is some (strictly) positive constant.

Given $f \in L^2(\Omega)$, we will regard $T_h f$ as a function in S_h that approximates the solution Tf of an elliptic differential equation problem.

We now make some observations concerning the $\{T_h(t)\}$ and $\{L_h(t)\}$ operators. Let $P : L^2(\Omega) \to S_h$ denote the orthogonal $L^2(\Omega)$-projection onto S_h and let

$Q \equiv I - P$. Then $T_h = PT_hP$, $T_h^* = PT_h^*P$ and (3.1)(iii) implies that the operator $T_h^{(\ell)}(t) \equiv \left(\frac{d}{dt}\right)^{\ell} T_h(t) : L^2(\Omega) \to S_h$ exists for any $\ell \geq 0$. We let $\bar{L}_h(t) \equiv \frac{1}{2}(L_h(t) + L_h^*(t))$ and $\bar{T}_h(t) \equiv (\bar{L}_h(t))^{-1}P$, for $0 \leq t \leq \tau$ and note that these selfadjoint operators are smooth in time and satisfy analogues of (3.1). Since $(L_h\varphi,\varphi) = (\bar{L}_h\varphi,\varphi)$ for $\varphi \in S_h$, we also see that

$$\|\varphi\|^2 \leq C(\overset{(-)}{L_h}\varphi,\varphi) \leq C\|\overset{(-)}{L_h}\varphi\|\,\|\varphi\| \quad \text{for all} \quad \varphi \in S_h \quad .$$

Thus, $\|T_h(t)\| \leq C$ and $\|\bar{T}_h(t)\| \leq C$ for $0 \leq t \leq \tau$.

We now use the family $\{T_h(t)\}$ to define an approximation to the solution u of (1.1). Choose a $v_h \in S_h$ (which should be thought of as an approximation to the initial data function $v \in L^2(\Omega)$) and let $u_h \in C^1([0,\tau],S_h)$ be the solution of

$$u_{h,t} + L_hu_h = 0 \quad \text{for} \quad 0 \leq t \leq \tau \quad \text{and} \quad u_h(0) = v_h \quad , \tag{3.2}$$

or equivalently,

$$T_hu_{h,t} + u_h = 0 \quad \text{for} \quad 0 \leq t \leq \tau \quad \text{and} \quad u_h(0) = v_h \quad . \tag{3.3}$$

The function $u_h(t)$ is our semidiscrete approximation for $u(t)$. A possible choice for v_h might be Pv. We will discuss other possibilities in a later Section.

We now study some properties of the solutions to equations like (3.2). We will include estimates of the time derivatives of such solutions that are independent of the dimension of S_h. To enable us to obtain these estimates, we will assume, throughout this Section, that the following hold for $0 \leq s,t \leq \tau$:

$$B_h \begin{cases} \|L_h^{(\ell)}(t)T_h(s)\|,\ \|T_h(t)L_h^{(\ell)}(s)P\| \leq C_B(\ell) \quad \text{for} \quad \ell > 0 \quad , \\ \|G_h^{(\ell)}(t)\varphi\|^2 \leq C_B(\ell)(L_h(t)\varphi,\varphi) \quad \text{for} \quad \ell > 0 \quad \text{and} \quad \varphi \in S_h \quad , \end{cases}$$

where $G_h^{(\ell)} \equiv \left(\frac{d}{dt}\right)^{\ell} G_h$. Straightforward calculations show that Condition B_h then holds for the $\{\bar{L}_h(t)\}$ family with, perhaps, new constants.

We note that (3.1) and Condition B_h imply that we can use the work of Sobolevskii [10] to study a fundamental solution operator for (3.2) and be assured of

dimension independent estimates. Thus, there is a family of operators $U_h(t,s)$ on S_h, defined for $0 \le s \le t \le \tau$, that is smooth in s and t for all $0 \le s \le t \le \tau$ and that satisfies the following for $0 \le s \le \xi \le t \le \tau$:

$$\text{(3.4)} \qquad \begin{cases} U_{h,t} + L_h(t)U_h = 0 \quad , \quad U_{h,s} - U_h L_h(s) = 0 \quad , \\ U_h(s,s) = I \quad , \quad U_h(t,\xi)U_h(\xi,s) = U_h(t,s) \quad . \end{cases}$$

We will write $U_h^{(m)}(t,s) \equiv \left(\frac{\partial}{\partial t}\right)^m U_h(t,s)$ for $0 \le s \le t \le \tau$ and $m \ge 0$. If $\hat{w}_h \in S_h$ and $f_h \in C([0,\tau],S_h)$, then the following generalization of (3.2):

$$\text{(3.5)} \qquad w_{h,t} + L_h w_h = f_h \quad \text{for} \quad 0 \le t \le \tau \quad \text{and} \quad w_h(0) = \hat{w}_h \quad ,$$

has an unique solution $w_h \in C^1([0,\tau],S_h)$ given by

$$\text{(3.6)} \qquad w_h(t) = U_h(t,0)\hat{w}_h + \int_0^t U_h(t,s)f_h(s)ds \quad .$$

If $m \ge 0$ and $0 \le s < t \le \tau$, we also have that

$$\text{(3.7)} \qquad \| U_h^{(m)}(t,s)P \| \le C(t-s)^{-m} \quad .$$

Versions of (2.17) and (2.18) hold for U_h and lead to certain estimates. For instance, if $0 \le t \le \tau$, $m \ge 0$ and $\varphi \in S_h$, we have that

$$\text{(3.8)(i)} \qquad U_h^{(m+1)}(t,0) + L_h(t)U_h^{(m)}(t,0) = -\sum_{\ell=0}^{m-1} \binom{m}{\ell} L_h^{(m-\ell)}(t)U_h^{(\ell)}(t,0) \quad \text{on} \quad S_h \quad ,$$

$$\text{(3.8)(ii)} \qquad \| L_h(t)U_h^{(m)}(t,0)\varphi \| \le C \sum_{\ell=0}^{m+1} \| U_h^{(\ell)}(t,0)\varphi \| \quad .$$

We now define operators $A_h^{(m)}(t)$ on S_h, for each $0 \le t \le \tau$ and $m \ge 0$, that satisfy $U_h^{(m)} = A_h^{(m)}U_h$. Let $A_h^{(0)} = I$ and, for $m \ge 0$, let

$$\text{(3.9)} \qquad \begin{aligned} A_h^{(m+1)}(t) &\equiv -\sum_{\ell=0}^{m} \binom{m}{\ell} L_h^{(m-\ell)}(t)A_h^{(\ell)}(t) \\ &= -A_h^{(m)}(t)L_h(t) + (A_h^{(m)}(t))_t \end{aligned}$$

(where the alternate characterization follows from the observation that $U_h^{(m+1)} = (U_h^{(m)})_t$ and a soon to be noted proof of the invertibility of U_h). We also have that

$$A_h^{(m)}(t) = (-L_h(t))^m + R_h^{(m)}(t)$$

where $R_h^{(\cdot)}(t)$ is a linear combination of operators that are at most an $(m-1)$-fold product of $L_h^{(\cdot)}$-operators, including at least one time differentiated operator.

If we let $\tilde{L}_h(t) \equiv L_h^*(\tau - t)$ for $0 \le t \le \tau$, we see that these operators satisfy all the assumptions we made on the family $\{L_h(t)\}$, with the same constants. We can thus define an associated fundamental solution operator $\tilde{U}_h$ and time differentiation operators $\{\tilde{A}_h^{(m)}\}$. We also have that

(3.10) $$U_h^*(t,s) = \tilde{U}_h(\tau - s, \tau - t) \quad \text{for} \quad 0 \le s \le t \le \tau \ .$$

We can also use the energy techniques of Section II to derive estimates. Let $\bar{w}_h(t) \equiv U_h(t,s)\bar{w}_h(s)$ for $s \le t \le \tau$ where $0 \le s < \tau$ is fixed and $\bar{w}_h(s) \in S_h$ and let $K_h = \sup_{0 \le \xi \le \tau} \|L_h(\xi)\|$. Then since

$$0 = \frac{1}{2}\frac{d}{dt}\|\bar{w}_h(t)\|^2 + (L_h\bar{w}_h, \bar{w}_h)(t) \le ((\frac{1}{2}\frac{d}{dt} + K_h)\|\bar{w}_h\|^2)(t) \quad \text{for} \quad s < t \le \tau \ ,$$

we can conclude that $U_h^{-1}(t,s) : S_h \to S_h$ exists for $0 \le s \le t \le \tau$. We can also provide an S_h-analogue of Proposition (2.2). In fact, if $w_h \in C^1([0,\tau],S_h)$ solves (3.5) where $w_h(0) = \hat{w}_h = 0$ and $f_h \in C^1([0,\tau],S_h)$, we have the following for $\epsilon > 0$:

(3.11) $$\|w_h(t)\| \le C \sup_{0 \le s \le t} \|(T_hf_h)(s)\| + \epsilon \sup_{0 \le s \le t} \|(T_hf_h)_s(s)\| \quad \text{for} \quad 0 \le t \le \tau \ .$$

To prove this result, we first note that S_h-analogues of (2.7) through (2.10) hold and that, because of Condition B_h, the following holds for every $\epsilon_1 > 0$:

(3.12) $$\begin{aligned} |(\bar{T}_h^{(1)}w_h, w_h)| &= |(\bar{L}_h^{(1)}\bar{T}_hw_h, \bar{T}_hw_h)| \\ &\le \epsilon_1(\bar{T}_h\bar{L}_h^{(1)}\bar{T}_hw_h, \bar{L}_h^{(1)}\bar{T}_hw_h) + C(\bar{T}_hw_h, w_h) \\ &\le C\epsilon_1\|w_h\|^2 + C(\bar{T}_hw_h, w_h) \ . \end{aligned}$$

Then since analogues of (2.12) through (2.16) hold, we can prove (3.11). The proof of Proposition (2.4) and the invertibility of the U_h operator than shows that if $m \geq 0$,

$$(3.13) \qquad (t-s)^{\ell} \| U_h^{(m)}(t,s)\varphi \| \leq \sum_{j=0}^{m-\ell} \| A_h^{(j)}(s)\varphi \|$$

for $0 \leq s \leq t \leq \tau$, $0 \leq \ell \leq m$ and $\varphi \in S_h$.

Thus we have (dimension independent) estimates for the solution of (3.2) (and (3.5)).

Now let $K \geq 0$ and set $L_{h,+}(t) \equiv L_h(t) + K$ on S_h and $T_{h,+}(t) \equiv (L_{h,+}(t))^{-1}P$ on $L^2(\Omega)$, for $0 \leq t \leq \tau$. If we define operators $A_{h,+}^{(m)}(t)$ for $m \geq 0$ and $0 \leq t \leq \tau$ using the S_h-analogue of (2.24), we see that an analogue of (2.25) holds in S_h. Also, the invertibility of U_h shows that an analogue of (2.26) holds in S_h. Finally, we can use the techniques introduced in the proof of Proposition (2.7) to show that an analogue of Proposition (2.7) is valid in S_h. If $m \geq 0$ and $K \geq 0$ is sufficiently large, there are invertible operators $\{E_{h,+}^{(\ell)}(t)\}_{\ell=0}^{m}$ on S_h for $0 \leq t \leq \tau$, given by an appropriate modification of (2.27), that satisfy

$$(3.14) \qquad A_{h,+}^{(m)}(t) = (-1)^m E_{h,+}^{(m)}(t) \cdots E_{h,+}^{(1)}(t) \quad \text{for} \quad 0 \leq t \leq \tau \ .$$

We let $A_{h,+}^{-(m)}(t) \equiv (A_{h,+}^{(m)}(t))^{-1}$ and note that $\| E_{h,+}^{-(m)}(t)P \| \leq C$ for $0 \leq t \leq \tau$.

IV. Error Estimates

We will now study how well the semidiscrete approximation $u_h(t)$ approximates the solution $u(t)$ of (1.1), given various conditions. We will continue to assume that the family $\{T_h(t)\}$ satisfies (3.1) but we will no longer assume that Condition B_h necessarily holds. We recall that the semidiscrete approximation u_h was defined in (3.2) (or (3.3)), where $v_h = u_h(0) \in S_h$ is thought of as an approximation for v. We now let $e(t) \equiv u_h(t) - u(t)$ and $\rho(t) \equiv (T - T_h)u^{(1)}(t)$ for $0 < t \leq \tau$. Note that

$$T_h e_t + e = \rho \quad \text{for} \quad 0 < t \leq \tau \quad . \tag{4.1}$$

We will analyze this error equation in a manner suggested by work in [3].

Our main estimate is given by the following:

Proposition (4.1): Suppose that e and $\rho \in C^1([0,\tau], L^2(\Omega))$ satisfy (4.1) and that for any $0 < \delta < 1$, there is a $C = C(\delta) \geq 0$ so that

$$|(\bar{T}_h^{(1)}(t)g,g)| \leq \delta\|g\|^2 + C(\delta)(\bar{T}_h(t)g,g) \quad \text{for} \quad g \in L^2(\Omega) \quad \text{and} \quad 0 \leq t \leq \tau \quad . \tag{4.2}$$

Choose $p = 0$ or 1 and if $p = 1$, suppose that Condition B_h holds. Then for any $\epsilon > 0$ and $0 \leq t \leq \tau$, we have that

$$t^p\|e(t)\| \leq C\|(T_h^p Pe)(0)\| + C(p \sup_{0\leq s\leq t}\|\sigma(s)\| + \sup_{0\leq s\leq t} s^p\|\rho(s)\| + \epsilon \sup_{0\leq s\leq t} s^{p+1}\|\rho_s(s)\|) \quad , \tag{4.3}$$

where $\sigma(t) = \int_0^t \rho(s)ds$ for $0 \leq t \leq \tau$.

We note that (3.12) shows that Condition B_h actually implies (4.2).

Proof: Let $w(s) = s^{m/2}e(s)$ where $m = 0$ if $p = 0$ or $m = 3$ if $p = 1$. Then

$$T_h w_s + w = s^{m/2}\rho + \frac{m}{2}s^{(m-2)/2}T_h e \quad \text{for} \quad 0 < s \leq \tau \quad , \tag{4.4}$$

where $w(0)$ is $e(0)$ if $m = 0$ or 0 if $m = 3$. Let $a(t) \in C^1([0,\tau], S_h)$ be defined by

$$T_h a_s + a = \frac{m}{2}s^{(m-2)/2}T_h e \; ; \; a(0) = \begin{cases} Pe(0) & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m = 3 \end{cases} \quad . \tag{4.5}$$

Thus if $w \equiv a + b$, then

(4.6) $$T_h b_s + b = s^{m/2}\rho \ ; \ b(0) = \begin{cases} Qe(0) & \text{if } m = 0 \\ 0 & \text{if } m = 3 \end{cases} .$$

Note that $s^{m/2}\rho - b \in S_h$ and that

(4.7) $$Pb_s = L_h(s^{m/2}\rho - b) = \bar{L}_h(s^{m/2}\rho - b) + G_h(s^{m/2}\rho - b) .$$

Thus, if we extend G_h to an operator on $L^2(\Omega)$ by the formula $G_h \equiv G_h P$, we have that

(4.8) $$\bar{T}_h b_s + b = s^{m/2}\rho + s^{m/2}\bar{T}_h G_h \rho - \bar{T}_h G_h b, \quad \text{for } 0 < s \leq \tau .$$

Let $0 < t \leq \tau$ and $\epsilon_1 > 0$. We can use (4.2), (4.6) and (4.8) and the fact that $(\bar{T}_h b, b)(0) = 0$ to show, as we did in the proof of Proposition (2.2), that

(4.9) $$\|b(t)\|^2 \leq C\Big(\sup_{0 \leq s \leq t} s^m \|\rho(s)\|^2 + \epsilon_1 \sup_{0 \leq s \leq t} s^{m+2} \|\rho_s(s)\|^2 \Big) .$$

Thus if $p = 0$ (so $m = 0$), the fact that $\|a(t)\| \leq \|Pe(0)\|$ allows us to complete the proof.

If $p = 1$ (so $m = 3$), we use (4.5) to see that

(4.10) $$(T_h a_s, a_s) + \frac{1}{2}\frac{d}{ds}\|a\|^2 = \frac{m}{2} s^{(m-2)/2}(T_h e, a_s) \quad \text{for } 0 < s \leq \tau .$$

We now use our usual techniques to show that if $\epsilon_2 > 0$, then

$$\begin{aligned} s^{\frac{m-2}{2}}(T_h e, a_s) &= s^{\frac{m-2}{2}}(L_h^* T_h e, T_h a_s) \\ &= s^{\frac{m-2}{2}}(\bar{L}_h T_h e, T_h a_s) - s^{\frac{m-2}{2}}(G_h T_h e, T_h a_s) \\ &\leq \epsilon_2(\bar{L}_h T_h a_s, T_h a_s) + Cs^{m-2}(\bar{L}_h T_h e, T_h e) \\ &\quad + \epsilon_2\|T_h a_s\|^2 + Cs^{m-2}\|G_h T_h e\|^2 \\ &\leq C\epsilon_2(T_h a_s, a_s) + Cs^{m-2}(T_h e, e) \quad \text{for } 0 < s \leq \tau , \end{aligned}$$

since $\|T_h\varphi\|^2 \le C(L_hT_h\varphi, T_h\varphi) = C(T_h\varphi,\varphi)$ for $\varphi \in S_h$ and

$$\|G_hT_h\varphi\|^2 \le C(L_hT_h\varphi,T_h\varphi) = C(T_h\varphi,\varphi) \quad \text{for} \quad \varphi \in S_h \quad .$$

Thus, we can show the following:

$$\|a(t)\|^2 \le C\int_0^t s(e,T_he)\,ds \quad . \tag{4.11}$$

Since the error equation implies that

$$(T_he_s,T_he) + (e,T_he) = \frac{1}{2}\frac{d}{ds}\|T_he\|^2 + (e,T_he) - (T_h^{(1)}e,T_he)$$

$$= (\rho,T_he) \quad \text{for} \quad 0 < s \le \tau$$

and since (using Condition B_h) we have that $(T_h^{(1)}e,T_he) \le C\|T_he\|^2$, we have the following:

$$\int_0^t s(e,T_he)\,ds \le C\int_0^t \|T_he\|^2 ds + C\int_0^t s^2\|\rho\|^2 ds \quad . \tag{4.12}$$

We now note that $T_he_s + e = (T_he)_s + e - T_h^{(1)}e = \rho$ for $0 < s \le \tau$, which implies that

$$T_he + \int_0^t e\,ds = \sigma + \int_0^t T_h^{(1)}e\,ds + (T_he)(0) \quad . \tag{4.13}$$

Thus it follows that

$$\|T_he\|^2 + \frac{d}{dt}\left(\int_0^t e, \bar{T}_h\int_0^t e\right) + \left\|\int_0^t e\right\|^2 = \left(\sigma + \int_0^t T_h^{(1)}e + (T_he)(0),\ T_he + \int_0^t e\right)$$

$$+ 2\left(\int_0^t e,(\bar{T}_h - T_h)e\right) + \left(\int_0^t e,\bar{T}_h^{(1)}\int_0^t e\right)$$

If $\epsilon_3 > 0$, we can use Condition B_h to see that

$$\left(\sigma + \int_0^t T_h^{(1)}e + (T_he)(0), T_he + \int_0^t e\right) \le \epsilon_3\|T_he\|^2 + \epsilon_3\left\|\int_0^t e\right\|^2$$

$$+ C\|\sigma\|^2 + C\|(T_he)(0)\|^2 + C\int_0^t \|T_he\|^2 ds$$

and we can (again) use an estimate suggested by (2.13) to see that

$$(\int_0^t e, (\bar{T}_h - T_h)e) = (\int_0^t e, \bar{T}_h G_h T_h e)$$

$$\leq C(\int_0^t e, \bar{T}_h(t) \int_0^t e) + \epsilon_3(\bar{T}_h G_h T_h e, G_h T_h e)$$

$$\leq C(\int_0^t e, \bar{T}_h \int_0^t e) + C\epsilon_3 \| T_h e \|^2 \quad .$$

Thus, if ϵ_3 is sufficiently small, we find that

$$\int_0^t \| T_h e \|^2 ds \leq C \int_0^t \| \sigma \|^2 ds + Ct \| T_h(0) e(0) \|^2 \quad . \tag{4.14}$$

We now use (4.9) through (4.14) to complete the proof.

Thus we see that if we want to estimate $\| u - u_h \|$ under various assumptions on v, it suffices to estimate quantities involving $\rho = (T - T_h)u_t$ and its derivatives.

We will assume that the following estimates hold throughout the remainder of this Section:

$$A_h \begin{cases} \text{There is an } r \geq 2 \text{ so that if } g \in H^\ell \text{ for some} \\ 0 \leq \ell \leq r-2 \text{ and } p \geq 0, \text{ then} \\ \| (T_h^{(p)}(t) - T^{(p)}(t))g \| \leq C_A(p) h^{\ell+2} \| g \|_\ell \text{ for } 0 \leq t \leq \tau \quad . \end{cases}$$

If we set $p = 0$ in the above inequality, then we are stating the usual kind of approximation assumption for Galerkin-type methods. We will show in a later Section that the inequalities are also reasonable for such methods if $p > 0$.

Our first application of Proposition (4.1) will be a preliminary result for smooth and compatible data v. We will need Condition B_h, described in Section III, if we wish to analyze the convergence of time derivatives.

Proposition (4.2): Let $m \geq 0$ and suppose that we have one of the following if $m = 0$:

(i) Condition A_h holds for the $\{\bar{T}_h\}$ and $\{\bar{T}\}$ families and $h > 0$ is sufficiently small,

(ii) The following estimate holds for some $C' \geq 0$:

(4.15) $\quad |(L_h^{(1)}(t)\varphi,\varphi)| \leq C'|(L_h(t)\varphi,\varphi)|$ for all $0 \leq t \leq \tau$ and $\varphi \in S_h$.

or (iii) Condition B_h holds.

If $m > 0$, assume that Condition B_h holds. Fix $0 \leq \ell \leq r-2$ and suppose that $v \in D(A^{(\alpha)}(0))$ where $\alpha = m + \frac{\ell+2}{2}$. Then

$$\| u^{(m)}(t) - u_h^{(m)}(t)\| \leq Ch^{\ell+2}\| v\|_{\ell+2+2m} \tag{4.16}$$
$$+ C\sum_{j=0}^{m} \| A_h^{(j)}(0)v_h - PA^{(j)}(0)v\| \quad .$$

<u>Proof</u>: We will use Proposition (4.1) to obtain this result. We first note that if we have Condition A_h for the barred families, then

$$(\bar{T}_h^{(1)}g,g) \leq Ch^2\| g\|^2 + (\bar{T}^{(1)}g,g) \leq Ch^2\| g\|^2 + C(\bar{T}g,g)$$
$$\leq Ch^2\| g\|^2 + C(\bar{T}_h g,g) \quad \text{for} \quad g \in L^2(\Omega) \quad \text{and} \quad 0 \leq t \leq \tau \quad .$$

Thus (4.2) would hold if h was small. If (4.15) holds, then

$$|(\bar{T}_h^{(1)}g,g)| \leq C'|(\bar{L}_h\bar{T}_h g,\bar{T}_h g)| = C'(\bar{T}_h g,g) \quad \text{for} \quad g \in L^2(\Omega) \quad \text{and} \quad 0 \leq t \leq \tau \quad .$$

Finally, if Condition B_h holds, (3.12) implies (4.2). Thus the hypotheses of this Proposition imply (4.2).

We now observe that if $0 < t \leq \tau$, then

$$\|\rho(t)\| \leq Ch^{\ell+2}\| u^{(1)}(t)\|_\ell \leq Ch^{\ell+2}\| u(t)\|_{\ell+2} \leq Ch^{\ell+2}\| v\|_{\ell+2} \quad ,$$

$$t\|\rho_t(t)\| \leq t\|(T^{(1)} - T_h^{(1)})u_t\| + t\|(T - T_h)u_{tt}\| \leq Ch^{\ell+2}\| v\|_{\ell+2} \quad .$$

Thus Proposition (4.1) implies that $\| e(t)\| \leq Ch^{\ell+2}\| v\|_{\ell+2} + C\| Pe(s)\|$ for any $0 < s \leq t \leq \tau$. We can now let $s \to 0$ to obtain our result for $m = 0$. Since $v \in D_L$, we also find that

$$\| T_h e_t\| \leq Ch^{\ell+2}\| v\|_{\ell+2} + C\| v_h - Pv\| \quad \text{for} \quad 0 \leq t \leq \tau \quad .$$

We now proceed by induction. We assume that $m \geq 1$, that $v \in D(A^{(\alpha)}(0))$ where $\alpha = m + \frac{\ell+2}{2}$ and that

$$(4.17) \qquad \|T_h e^{(p+1)}\| \leq Ch^{\ell+2}\|v\|_{\ell+2+2p} + C\sum_{j=0}^{p} \|A_h^{(j)}(0)v_h - PA^{(j)}(0)v\|$$

for $0 \leq p \leq m-1$ and $0 < t \leq \tau$. Since $v \in D(A^{(m+1)}(0))$, we can differentiate the error equation and show for $0 < t \leq \tau$, that

$$(4.18) \qquad T_h e_t^{(m)} + e^{(m)} = \rho^{(m)} - mT_h^{(1)}e^{(m)} - \sum_{j=0}^{m-2} \binom{m}{j} T_h^{(m-j)} e^{(j+1)} \quad .$$

We now apply Proposition (4.1) in the manner described before and show that the following holds, for $\epsilon > 0$:

$$(4.19) \qquad \|e^{(m)}(t)\| \leq Ch^{\ell+2}\|v\|_{\ell+2+2m} + C\epsilon \sup_{0\leq s\leq t} \|T_h e^{(m+1)}(s)\|$$

$$+ C\sum_{j=0}^{m} \|A_h^{(j)}(0)v_h - PA^{(j)}(0)v\|$$

$$\leq Ch^{\ell+2}\|v\|_{\ell+2+2m} + \frac{1}{2}\sup_{0\leq s\leq t}\|e^{(m)}(s)\|$$

$$+ C\sum_{j=0}^{m} \|A_h^{(j)}(0)v_h - PA^{(j)}(0)v\| \quad .$$

The estimate given by (4.19) implies (4.16) and we can return to (4.18) to complete the induction step. This completes the proof.

We will now describe choices for v_h that will obtain $O(h^r)$ convergence for the error and some of its derivatives if v is sufficiently smooth and compatible. The description will be easiest if we only wish to describe the error and one time derivative.

Before we give this first result, we make an observation concerning the map $P_1(t) \equiv T_h(t)L(t) : D_L \to S_h$ defined for $0 \leq t \leq \tau$, often known as the "elliptic projection." By our approximation assumption A_h, we know that

$$\|w - P_1 w\| = \|(T - T_h)Lw\| \le Ch^{\ell+2}\|w\|_{\ell+2} \quad \text{for} \quad 0 \le t \le \tau \quad ,$$

if $w \in H^{\ell+2} \cap D_L$ for some $0 \le \ell \le r - 2$ and hence that

$$\|w - Pw\| \le \|w - P_1 w\| \le Ch^{\ell+2}\|w\|_{\ell+2} \quad \text{for} \quad 0 \le t \le \tau \quad .$$

Corollary (4.3): (1) Suppose that $v \in D(A^{\left(\frac{\ell+2}{2}\right)}(0))(\subset H^{\ell+2} \cap D_L)$ for some $0 \le \ell \le r - 2$ and either (i), (ii) or (iii) of Proposition (4.2) holds. Then if $v_h = Pv$ or $v_h = P_1(0)v$ or $v_h \in S_h$ is chosen so that $\|v_h - v\| \le Ch^{\ell+2}\|v\|_{\ell+2}$, we have that

$$\|u(t) - u_h(t)\| \le Ch^{\ell+2}\|v\|_{\ell+2} \quad \text{for} \quad 0 \le t \le \tau \quad . \tag{4.20}$$

(2) Suppose that Condition B_h holds and $v \in D(A^{\left(\frac{\ell+4}{2}\right)}(0))$ for some $0 \le \ell \le r - 2$. Then if $v_h = P_1(0)v$ or $v_h = T_h^2(0)L^2(0)v$ or $v_h \in S_h$ is chosen so that $\|L_h(0)v_h - L(0)v\| \le Ch^{\ell+2}\|v\|_{\ell+4}$, we have that

$$\|u(t) - u_h(t)\| + \|u^{(1)}(t) - u_h^{(1)}(t)\| \le Ch^{\ell+2}\|v\|_{\ell+4} \quad \text{for} \quad 0 \le t \le \tau \quad . \tag{4.21}$$

Proof: Part (1) follows easily from Proposition (4.2). As for Part (2), we see that

$$\|v_h - P_1 v\| \le C\|L_h(0)(v_h - P_1 v)\| \le C(\|L_h(0)v_h - L(0)v\| + \|(P - I)L(0)v\|) \quad .$$

The result now easily follows from Proposition (4.2).

We now wish to study approximation results for higher time derivatives. To describe and prove such results, we first need to study some properties of the $\{A_+^{(j)}(0)\}_{j\ge 0}$ and $\{A_{h,+}^{(j)}(0)\}_{j\ge 0}$ operators. We have the following analogues of Condition A_h.

Proposition (4.4): Let $K \ge 0$ and suppose that Condition B_h holds. Then, if $g \in H^{\ell}$ for some $0 \le \ell \le r - 2$, we have that

(4.22)(i) $\quad \|(T_{h,+}^{(p)}(t) - T_{+}^{(p)}(t))g\| \le C(p)h^{\ell+2}\|g\|_{\ell}$ for $0 \le t \le \tau$ and $p \ge 0$;

that is, we have Condition A_h for the $\{T_{+}(t)\}$ and $\{T_{h,+}(t)\}$ families. Moreover, if K is sufficiently large, we also have that

(4.22)(ii) $\quad \|(E_{h,+}^{-(p)}(t)P - E_{+}^{-(p)}(t))g\| \le Ch^{\ell+2}\|g\|_{\ell}$ for $0 \le t \le \tau$ and $p \ge 1$.

Proof: If $K \ge 0$, $0 \le t \le \tau$ and $g \in H^{\ell}$ for some $0 \le \ell \le r - 2$, then $g = L_{+}T_{+}g = LT_{+}g + KT_{+}g$. Thus, $T_{+}g = Tg - KT\,T_{+}g$ and similarly, $T_{h,+}g = T_h g - KT_h T_{h,+}g$. Let $p \ge 0$, set $E \equiv (T_{+}^{(p)} - T_{h,+}^{(p)})g$ and suppose we know that $\|(T_{+}^{(j)} - T_{h,+}^{(j)})f\| \le Ch^{\ell+2}\|f\|_{\ell}$ if $0 \le j \le p - 1$ and $f \in H^{\ell}$. Then, since

$$E = (T^{(p)} - T_h^{(p)})g + K\sum_{\ell=0}^{p-1}\binom{p}{\ell} T_h^{(p-\ell)}(T_{h,+}^{(\ell)} - T_{+}^{(\ell)})g$$

$$- KT_hE + K\sum_{\ell=0}^{p}\binom{p}{\ell}(T_h^{(p-\ell)} - T^{(p-\ell)})T_{+}^{(\ell)}g \quad ,$$

taking the $L^2(\Omega)$-inner product of the above with E shows that

$$\|E\|^2 \le C(K)h^{\ell+2}\|g\|_{\ell}\,\|E\| - K(T_hE,E) \le C(K)h^{\ell+2}\|g\|_{\ell}\,\|E\| \quad .$$

We can now prove (4.22)(i) by induction.

We will also prove (4.22)(ii) by induction. We know that (4.22)(ii) holds with $p = 1$. We now suppose we have (4.22)(ii) for some $p \ge 1$ and all the intermediate cases. Let $0 \le t \le \tau$ and $g \in H^{\ell}$ for some $0 \le \ell \le r - 2$. If $K \ge 0$ is sufficiently large, we can set $w \equiv E_{+}^{-(p+1)}g$, $w_h \equiv E_{+}^{-(p+1)}Pg$ and

$$R \equiv \sum_{j=0}^{p-2}(-1)^{p-j+1}\binom{p}{j} L_{+}T_{+}^{(p-j)}E_{+}^{-(j+2)} \cdots E_{+}^{-(p)}$$

and let R_h be R's counterpart on S_h. Note that

$$w = T_{+}g - p\,T_{+}^{(1)}w - T_{+}R_{+}w \quad \text{and} \quad w_h = T_{h,+}g - p\,T_{h,+}^{(1)}w_h - T_{h,+}R_hw_h \quad .$$

Thus if K is sufficiently large, we have that

$$\|w - w_h\| \le C[\|(T_+ - T_{h,+})g\| + \|(T_+^{(1)} - T_{h,+}^{(1)})w\|$$

$$+ \|(T_+R - T_{h,+}R_hP)w\| + \|T_{h,+}^{(1)}(w - w_h)\| + \|T_{h,+}R_hP(w - w_h)\|]$$

$$\le C(K)h^{\ell+2}\|g\|_\ell + \frac{1}{2}\|w - w_h\| + \|(T_+R - T_{h,+}R_hP)w\| \quad ,$$

$$\|(T_+R - T_{h,+}R_hP)w\| \le C \sum_{j=0}^{p-2} [\|(T_+^{(p-j)} - T_{h,+}^{(p-j)})E_+^{-(j+2)} \cdots E_+^{-(p)}w\|$$

$$+ \|\sum_{i=j+2}^{p} A_{h,+}^{(j+1)}A_{h,+}^{-(i-1)}P(E_{h,+}^{-(i)}P - E_+^{-(i)})A_+^{(i)}A_+^{-(p)}w\|]$$

$$\le C(K)h^{\ell+2}\|w\|_{\ell+2} \le C(K)h^{\ell+2}\|g\|_\ell \quad .$$

This gives (4.22)(ii) for $p + 1$, which gives the induction step.

We can now complete our estimates for higher derivatives of the error when v is a sufficiently smooth and compatible function.

<u>Theorem (4.5)</u>: Suppose that $m \ge 2$, Condition B_h holds and $v \in D(A^{(\alpha)}(0))$ where $\alpha = m + \frac{\ell+2}{2}$ and $0 \le \ell \le r - 2$. Let $K \ge 0$ be sufficiently large and choose $w_h \in S_h$ so that

$$(4.23) \qquad \|w_h - A_+^{(m)}(0)v\| \le Ch^{\ell+2}\|v\|_{\ell+2+2m} \quad ;$$

for instance, let $w_h = PA_+^{(m)}(0)v$ or $P_1(0)A_+^{(m)}(0)v$. Then if $v_h \equiv A_{h,+}^{-(m)}(0)w_h$, we have that

$$(4.24) \qquad \sum_{j=0}^{m} \|u^{(j)}(t) - u_h^{(j)}(t)\| \le Ch^{\ell+2}\|v\|_{\ell+2+2m} \quad \text{for } 0 \le t \le \tau \quad .$$

<u>Proof</u>: Proposition (4.2) and (2.26) and its S_h-analogue show that it suffices to know that

$$\|A_{h,+}^{(j)}A_{h,+}^{-(m)}w_h - PA_+^{(j)}A_+^{-(m)}A_+^{(m)}v\| \le Ch^{\ell+2}\|v\|_{\ell+2+2m} \quad ,$$

for each $0 \le j \le m - 1$. But (4.22) shows that the following holds for $0 \le j \le m - 1$ and $g \in H^\ell$:

$$\| A_{h,+}^{(j)} A_{h,+}^{-(m)} Pg - A_{+}^{(j)} A_{+}^{-(m)} g \| \leq C h^{\ell+2} \| g \|_{\ell} \quad .$$

Since $\| A_{h,+}^{(j)} A_{h,+}^{-(m)} P \| \leq C$ for $0 \leq j \leq m-1$ and

$$\| (P - I) A_{+}^{(j)} v \| \leq C h^{\ell+2} \| v \|_{\ell+2+2j} \quad \text{for} \quad 0 \leq j \leq m-1 \quad ,$$

we have our result.

We now will discuss convergence results for the error and its derivatives when v is no more than a function in $L^2(\Omega)$. We will limit our choice of initial data for the semidiscrete approximation to $v_h = Pv$ for these nonsmooth data results.

<u>Theorem (4.5)</u>: Suppose that Condition B_h holds. Then if $v_h = Pv$ and $m \geq 0$, we have that

$$(4.25) \qquad \sum_{j=0}^{m} \| u^{(j)}(t) - u_h^{(j)}(t) \| \leq C h^r t^{-r/2-m} \| v \| \quad \text{for} \quad 0 < t \leq \tau \quad .$$

<u>Proof</u>: We begin by noting that the estimates of Sections II and III and density arguments allow us to assume that $v \in C_c^{\infty}(\Omega)$. These estimates also show that we can assume that $h^2 \leq t$.

We now use Proposition (4.1) with $p = 1$. Since

$$\| \sigma(t) \| = \| \int_0^t (T_h - T)(s) u_s(s) ds \|$$

$$\leq \| (T_h - T) u(t) \| + \| (T_h - T) u(0) \| + \int_0^t \| (T_h^{(1)} - T^{(1)}) u \| ds$$

$$\leq C h^2 \| v \| \quad \text{for} \quad 0 \leq t \leq \tau \quad ,$$

and similarly, $t \| \rho(t) \| + t^2 \| \rho_t(t) \| \leq C h^2 \| v \|$ for $0 \leq t \leq \tau$, we see we have shown that

$$(4.26) \qquad \| u(t) - u_h(t) \| = \| (U\ t,0) - U_h(t,0) P) v \| \leq C h^2 t^{-1} \| v \|$$

for $0 < t \leq \tau$. Thus we have shown (4.25) for $m = 0$ and $r = 2$. To obtain (4.25) with $m \geq 1$, we will need to use this result and our previous smooth data results in a bootstrapping argument. We will also use a special representation for smooth and

compatible functions. (This representation was used in a slightly different context in [1]). Suppose that $p \geq 1$ and $w \in D(A^{(p)}(t))$ for some $0 \leq t \leq \tau$. Then, if $K \geq 0$ is sufficiently large,

$$w = A_{h,+}^{-(p)} P A_{+}^{(p)} w + (E_{+}^{-(1)} - E_{h,+}^{-(1)} P) E_{+}^{(1)} w$$
$$- \sum_{j=2}^{p} A_{h,+}^{-(j-1)} P (E_{+}^{-(j)} - E_{h,+}^{-(j)} P) A_{+}^{(j)} w \quad . \tag{4.27}$$

Let $0 \leq p \leq m$, $0 < h^2 \leq t \leq \tau$, $t_1 \equiv t/3$, $t_2 \equiv 2t/3$ and $v \in C_c^\infty(\Omega)$. Then

$$\|u^{(p)}(t) - u_h^{(p)}(t)\| = \|(U^{(p)}(t,0) - U_h^{(p)}(t,0)P)v\|$$
$$\leq \|(U^{(p)}(t,t_2) - U_h^{(p)}(t,t_2)P)U(t_2,0)v\|$$
$$+ \|U_h^{(p)}(t,t_2)P\| \, [\|(U(t_2,t_1) - U_h(t_2,t_1)P)U(t_1,0)v)\|$$
$$+ \|U(t_2,t_1)(U(t_1,0) - U_h(t_1,0)P)v\|$$
$$+ \|U(t_2,t_1) - U_h(t_2,t_1)P\| \; \|U(t_1,0) - U_h(t_1,0)P\| \; \|v\|] \quad .$$

$$\leq \text{term}_1 + Ct^{-p}(\text{term}_2 + \text{term}_3 + \text{term}_4) \quad .$$

By our smooth data results applied with a suitable translation of time zero, we find that if K is large, then

$$\text{term}_1 \leq \|U^{(p)} u(t_2) - U_h^{(p)} A_{h,+}^{-(p)} P A_{+}^{(p)} u(t_2)\|$$
$$+ \sum_{j=1}^{p} \|U_h^{(p)} A_{h,+}^{-(j-1)} P (E_{+}^{-(j)} - E_{h,+}^{-(j)} P) A_{+}^{(j)} u(t_2)\|$$
$$\leq Ch^r \|u(t_2)\|_{r+2p} + C \sum_{j=1}^{p} t^{-p-1+j} h^r \|u(t_2)\|_{r-2+2j}$$
$$\leq Ch^r t^{-r/2-p} \|v\| \quad .$$

We observed in (4.20) that $\text{term}_2 \leq Ch^r t^{-r/2} \|v\|$. Also,

$$\text{term}_3 \le \|U(U - U_h P)\| \; \|v\| = \|(U^* - U_h^* P)U^*\| \; \|v\|$$

$$= \|(\tilde{U} - \tilde{U}_h P)\tilde{U}\| \; \|v\| \le Ch^r t^{-r/2} \|v\|$$

where we have used our adjoint identifications and smooth data results applied to the time-reversed operators. Finally, $\text{term}_4 \le C(h^2 t^{-1})^2 \|v\|$ by (4.26).

By iterating the above argument μ times with $p = 0$ we find that

$$\|U(t,0) - U_h(t,0)P\| \le C((h^2 t^{-1})^{r/2} + (h^2 t^{-1})^{2^\mu}) \quad ,$$

for any $0 < t \le \tau$. We now choose $\mu \ge 1$ so that $2^{\mu+1} \ge r$. The proof is then easily completed by redoing the argument once for each $1 \le p \le m$.

We conclude this Section by noting the following result on forcing terms. If we assume we are given a suitably smooth function $u(x,t)$ on $\bar{\Omega} \times [0,\tau]$ that satisfies

$$u_t + L(t)u = f(t) \quad \text{on} \quad \Omega \times [0,\tau] \quad \text{and} \quad u(0) = v \quad \text{on} \quad \Omega \quad ,$$

where f is suitably smooth on $\bar{\Omega} \times [0,\tau]$, we can define a semidiscrete approximation by the following:

$$u_{h,t} + L_h u_h = Pf \quad \text{for} \quad 0 < t \le \tau \quad \text{and} \quad u_h(0) = v_h \quad ,$$

where v_h is chosen from S_h. (We note that $u_h(t)$ always exists). If we set $e \equiv u_h - u$, we find that

$$T_h e_t + e = (T - T_h)(u_t - f) = -(T - T_h)Lu \quad \text{for} \quad 0 < t \le \tau \quad .$$

This equation can be easily analyzed by Proposition (4.1). For instance, if we set $v_h = Pv$ and assume Condition B_h, then

$$\|u(t) - u_h(t)\| \le C(u)h^r$$

for some constant $C(u)$ depending on the solution.

V. Examples

We will summarize here some well known results concerning several [illegible]-projection methods and we will give the additional required estimates for these methods that will allow us to apply the theory of the preceding Sections. We begin by sketching some of the common features of the methods.

Each method will use a finite dimensional subspace S_h of functions in H^1_* and will be associated with parameters $0 < h < 1$ and an $r \geq 2$ in the following way:

$$(5.1) \qquad \min_{\varphi \in S_h} \left\{ \|w - \varphi\| + h\|w - \varphi\|_I \right\} \leq Ch^{\ell+2}\|w\|_{\ell+2}$$

for all $w \in H^{\ell+2} \cap H^1_*$, where $0 \leq \ell \leq r - 2$ and where $\|\cdot\|_I$ is a (perhaps h-dependent) seminorm that will be related to the $\|\cdot\|_1$ norm but which may contain other terms dealing with boundary condition considerations.

For each of our coercive differential operators $L(t)$, there will be an associated positive form $D_h(t)(\cdot,\cdot)$ that is related to the Dirichlet form of the operator and which will be used to define the associated $T_h(t)$ operator. Given $f \in L^2(\Omega)$, the function $w_h = T_h(t)f \in S_h$ will be defined by the following equations:

$$(5.2) \qquad D_h(t)(w_h,\varphi) = (f,\varphi) \quad \text{for} \quad \varphi \in S_h \quad .$$

The form $D_h(t)$ will be symmetric positive definite if $L = L^* = \bar{L}$.

The following relations will hold, with certain constants:

$$(5.3) \qquad C_1\|\varphi\|_1^2 \leq \|\varphi\|_I^2 \leq C_2 D_h(t)(\varphi,\varphi) \quad \text{for all} \quad \varphi \in S_h \quad ,$$

$$(5.4) \qquad |D_h^{(\ell)}(t)(g_1,g_2)| \leq C_3(\ell)\|g_1\|_I\|g_2\|_I \quad \text{for} \quad \ell \geq 0 \quad ,$$

where g_1 and $g_2 \in S_h + D_L$ and $D_h^{(\ell)}(t)(\cdot,\cdot)$ denotes the form obtained by taking ℓ successive time derivatives of the D_h-form. We will also have the following for $j \geq 0$:

$$(5.5)\qquad \left.\begin{array}{l} |(L^{(j)}w,g) - D_h^{(j)}(w,g)| \\ \\ |(g,L^{*(j)}w) - D_h^{(j)}(g,w)| \end{array}\right\} \le C_*(j)h^{\ell+1}\|w\|_{\ell+2}\|g - z\|_I$$

where $w \in H^{\ell+2} \cap D_L$ for some $0 \le \ell \le r-2$, $g \in S_h + D_L$ and $z \in D_L$ is arbitrary. (Thus we will "almost" be able to integrate by parts). Finally, we will have that

$$(5.6)\qquad G_h(t)\varphi = P\Big(\sum_{i=1}^{d} a_{0i}(t)D_i\varphi + \frac{1}{2}\Big(\sum_{i=1}^{d} D_i a_{0i}(t)\Big)\varphi\Big) \quad \text{for } \varphi \in S_h \;,$$

where $P : L^2(\Omega) \to S_h$ is the usual $L^2(\Omega)$-orthogonal projection.

We will now list what the $\|\cdot\|_I$-seminorm and the D_h-form is for each method. We will then go on to prove the necessary stability and accuracy results. (There are references given in [2] and [3] for the various methods. We will not repeat them here).

(1) Galerkin's Method for the Neumann Problem:

In this case $D_h(t) = D(t)$, $\|\cdot\|_I = \|\cdot\|_1$ and C_* can be taken to be 0.

(2) Galerkin's Method for the Dirichlet Problem:

Now we must restrict S_h to be in H_0^1. Thus boundary conditions are required of functions in S_h. Again $D_h(t) = D(t)$, $\|\cdot\|_I = \|\cdot\|_1$ and $C_* = 0$.

(3) A Method of Nitsche:

This is a technique for the Dirichlet problem that does not require that S_h lie in H_0^1. The norm and forms for the method are as follows:

$$D_h(t)(\cdot,\cdot) \equiv D(t)(\cdot,\cdot) - \langle \cdot, \frac{\partial}{\partial \underline{n}}(\cdot)\rangle - \langle \frac{\partial}{\partial \underline{n}}(\cdot),\cdot\rangle + \beta h^{-1}\langle \cdot,\cdot\rangle \;,$$

$$\|\cdot\|_I^2 = \|\cdot\|_1^2 + C_I h^{-1}\|\cdot\|_{0,\partial\Omega}^2 + C_{II}h[\|(\cdot)_{\underline{n}}\|_{0,\partial\Omega}^2 + \|\cdot\|_{1,\partial\Omega}^2] \;,$$

where $\frac{\partial}{\partial \underline{n}}$ denotes the conormal derivative associated with $\bar{L}(t)$, $(\cdot)_{\underline{n}}$

denotes the normal derivative, C_I and C_{II} are certain constants and $\beta \geq 0$ is sufficiently large. The following inverse assumption is required on S_h as well:

$$\|(\varphi)_{\underline{n}}\|_{0,\partial\Omega} + \|\varphi\|_{1,\partial\Omega} \leq C_{III}(h^{-\frac{1}{2}}\|\varphi\|_1 + h^{-1}\|\varphi\|_{0,\partial\Omega}) \quad \text{for all} \quad \varphi \in S_h \ .$$

We note that we can take $C_* = 0$.

(4) Another Method of Nitsche:

This is another method that handles the Dirichlet problem without requiring that S_h lie in H^1_o. The method is the same as the previous one but β may be taken to be zero. However the following "almost zero boundary conditions" restriction must be put on S_h :

$$\|\varphi\|_{0,\partial\Omega} \leq C_{IV}h^{\frac{1}{2}}\|\varphi\|_1 \quad \text{for all} \quad \varphi \in S_h$$

where C_{IV} is sufficiently small.

(5) A Lagrange Multiplier Method of Babuška:

This is yet another approach for approximating the solution of the Dirichlet problem without imposing boundary conditions on functions in S_h. The space S_h is constructed in a special way so as to agree with a Lagrange Multiplier formulation that would be used in practice. We will not detail this construction here, but note that the key to the boundary conditions is given by the following estimates. If $w \in H^{\ell+2} \cap H^1_o$ where $0 \leq \ell \leq r - 2$, $g \in S_h + D_L$ and $z \in D_L$ is arbitrary, then

$$\begin{aligned}|\langle \sum_{j=1}^{d} a_{ij}n_iD_jw,g\rangle| &= |\langle \sum_{j=1}^{d} a_{ij}n_iD_jw,g - z\rangle| \\ &\leq Ch^{\ell+1}\|w\|_{\ell+2}\|g - z\|_1 \quad \text{for} \quad 1 \leq i \leq d \ ,\end{aligned} \tag{5.7}$$

due to certain boundary approximation properties. We use $D_h(t) = D(t)$ and $\|\cdot\|_I = \|\cdot\|_1$ for this method and (5.5) follows from (5.7).

We will now prove estimates that will hold for any approximation method that is defined by (5.2) where (5.1) and (5.3) through (5.6) hold. We first see that

$$(5.8)\qquad \left|\sum_{i=0}^{j}\binom{j}{i} D_h^{(j-i)}(t)((T_h^{(i)}(t) - T^{(i)}(t))f,\varphi_2)\right| \le CC_*\|\varphi_2 - z\|_I h^{\ell+1}\|f\|_\ell \quad ,$$

$$(5.9)\qquad (L_h^{(j)}(t)\varphi_1,\varphi_2) = D_h^{(j)}(t)(\varphi_1,\varphi_2) = (\varphi_1,L_h^{*(j)}(t)\varphi_2) \quad ,$$

for $j \ge 0$, $0 \le t \le \tau$, $\varphi_1,\varphi_2 \in S_h$, $z \in D_L$ and $f \in H^\ell$, for some $0 \le \ell \le r - 2$.

The following is our main approximation result.

<u>Theorem (5.1)</u>: For $m \ge 0$, $0 \le \ell \le r - 2$ and $0 \le t \le \tau$, we have that

$$(5.10)\qquad \|(T_h^{(m)}(t) - T^{(m)}(t))f\|_I \le Ch^{\ell+1}\|f\|_\ell \quad \text{for } f \in H^\ell \quad ,$$

$$(5.11)\qquad \|(T_h^{(m)}(t) - T^{(m)}(t))f\| \le Ch^{\ell+2}\|f\|_\ell \quad \text{for } f \in H^\ell \quad .$$

<u>Proof</u>: We will first prove (5.10) and then we will use a duality argument to prove (5.11). Say $0 \le \ell \le r - 2$ and $f \in H^\ell$. Set $w^{(j)} \equiv T^{(j)}f$ and $w_h^{(j)} \equiv T_h^{(j)}f$ for $0 \le j \le m$. Let $\tilde{w}_h^{(j)} \in S_h$ be the function that attains the minimum in (5.1) when we replace w by $w^{(j)}$, for each $0 \le j \le m$. Then

$$C\|\tilde{w}_h^{(m)} - w_h^{(m)}\|_I^2 \le D_h(\tilde{w}_h^{(m)} - w_h^{(m)},\ \tilde{w}_h^{(m)} - w_h^{(m)})$$

$$\le D_h(\tilde{w}_h^{(m)} - w^{(m)},\ \tilde{w}_h^{(m)} - w_h^{(m)}) + CC_*\|\tilde{w}_h^{(m)} - w_h^{(m)}\|_I h^{\ell+1}\|f\|_\ell$$

$$+ \left|\sum_{j=0}^{m-1}\binom{m}{j} D_h^{(m-j)}(w_h^{(j)} - w^{(j)},\ \tilde{w}_h^{(m)} - w_h^{(m)})\right|$$

$$\le C\|\tilde{w}_h^{(m)} - w_h^{(m)}\|_I (Ch^{\ell+1}\|f\|_\ell + \sum_{j=0}^{m-1}\|w_h^{(j)} - w^{(j)}\|_I) \quad .$$

This can be used to show (5.10) for all $m \ge 0$.

Now we use a duality argument. Let $z \in D_L$ satisfy $L^*z = w^{(m)} - w_h^{(m)}$ and let $\tilde{z}_h \in S_h$ be the function that attains the minimum in (5.1) when we replace w by z. Then

$$\|w^{(m)} - w_h^{(m)}\|^2 = (w^{(m)} - w_h^{(m)}, L^*z)$$

$$= \sum_{j=0}^{m} \binom{m}{j} (w^{(j)} - w_h^{(j)}, L^{*(m-j)}z) - \sum_{j=0}^{m-1} \binom{m}{j} (w^{(j)} - w_h^{(j)}, L^{*(m-j)}z)$$

$$\leq \sum_{j=0}^{m} \binom{m}{j} D_h^{(m-j)}(w^{(j)} - w_h^{(j)}, z - \check{z}_h) + CC_* \sum_{j=0}^{m} \|w^{(j)} - w_h^{(j)}\|_I h\|z\|_2$$

$$+ \sum_{j=0}^{m} \binom{m}{j} D_h^{(m-j)}(w^{(j)} - w_h^{(j)}, \check{z}_h) + C \sum_{j=0}^{m-1} \|w^{(j)} - w_h^{(j)}\| \|z\|_2$$

$$\leq (Ch^{\ell+2}\|f\|_{\ell} + C \sum_{j=0}^{m-1} \|w^{(j)} - w_h^{(j)}\|)\|w^{(m)} - w_h^{(m)}\| .$$

This can be used to complete the proof.

We also have the following:

<u>Proposition (5.2)</u>: If $\ell \geq 0$ and $0 \leq t \leq \tau$ then

(5.12) $$\|G_h^{(\ell)}(t)\varphi\|^2 \leq C(L_h(t)\varphi,\varphi) = C(\bar{L}_h(t)\varphi,\varphi) \quad \text{for all } \varphi \in S_h .$$

<u>Proof</u>: We see that if $\varphi \in S_h$,

$$\|G_h^{(\ell)}(t)\varphi\|^2 \leq C\|\varphi\|_1^2 \leq C\|\varphi\|_I^2 \leq C(L_h(t)\varphi,\varphi) .$$

We will now study the remainder of Condition B_h, using inverse properties (as was also done in [7]).

<u>Proposition (5.3)</u>: Suppose that

(5.13) $$\|\varphi\|_I \leq Ch^{-1}\|\varphi\| \quad \text{for all } \varphi \in S_h .$$

Then for $\ell > 0$ and $0 \leq s, t \leq \tau$, we have that

(5.14) $$\|L_h^{(\ell)}(t)T_h(s)\| \quad , \quad \|T_h(t)L_h^{(\ell)}(s)P\| \leq C(\ell) .$$

<u>Proof</u>: If $g \in L^2(\Omega)$, $\varphi \in S_h$ and $\ell \geq 0$, then

$$|(L_h^{(\ell)}(t)T_h(s)g,\varphi)| = |D_h^{(\ell)}(t)((T_h(s) - T(s))g,\varphi)| + |D_h^{(\ell)}(t)(T(s)g,\varphi)|$$

$$\leq C\|(T_h(s) - T(s))g\|_I\|\varphi\|_I + |(L^{(\ell)}(t)T(s)g,\varphi)| + C_*h\|T(s)g\|_2\|\varphi\|_I$$

$$\leq C\|g\| \|\varphi\| .$$

This suffices to prove the first part of (5.14) and the rest of the proof follows by considering the adjoint problem.

We note that if we have (5.13), $\|L_h(t)P\| \leq Ch^{-2}$ for $0 \leq t \leq \tau$.

Thus we can now apply the analysis of Sections II through IV to many Galerkin-type projection methods.

VI. Maximum Norm Estimates

We will now examine how we can use our $L^2(\Omega)$-based estimates in conjunction with maximum norm estimates for the associated elliptic problem to prove similar maximum norm estimates for the parabolic problem. We will use techniques similar to those introduced in [3].

We first study *global maximum norm estimates*. *We assume*, as usual, that we have a family of operators $\{T_h(t)\}$ that satisfies the properties listed in (3.1) and the approximation assumptions A_h. We will also assume that we are working in $d = 1, 2$ or 3 space dimensions and that $S_h \subset C^{\delta}(\bar{\Omega})$ for some constant $0 < \delta \leq 1$; that is, the functions in S_h are Hölder continuous with exponent δ on $\bar{\Omega}$. Finally, we will assume the following inverse property on S_h :

$$(6.1) \qquad \|\varphi\|_{C^{\epsilon}(\Omega)} \leq C(\epsilon)h^{-d/2-\epsilon}\|\varphi\|, \quad \text{for } 0 < \epsilon \leq \delta \text{ and } \varphi \in S_h .$$

To obtain maximum norm estimates for the parabolic problem, we need to know some *corresponding* estimates for the elliptic problem. We will assume the following:

$$(6.2) \quad \begin{cases} \text{There is a function } \gamma_p(h) \text{ so that if } T(t)f \in W^{p,\infty} \text{ for some } 0 \leq t \leq \tau, \\ \text{we have that} \\ \qquad \|(T_h - T)(t)f\|_{0,\infty} \leq \gamma_p(h)\|T(t)f\|_{p,\infty} , \\ \text{where } p = 2 \text{ or } r . \end{cases}$$

We will also assume that $\gamma_2(h) \leq Ch^{2-\eta}$ *for some* $\eta < \frac{1}{2}$. Work in [4] or [9] done under various conditions suggests that we could take $\gamma_2(h) = Ch^2$, $\gamma_r(h) = Ch^r$ if $r > 2$ and $\gamma_2(h) = \gamma_r(h) = C|\log h|h^2$ if $r = 2$.

We note that these conditions imply the following:

$$\begin{aligned} \|T_h\varphi\|_{0,\infty} &\leq \|(T_h - T)\varphi\|_{0,\infty} + \|T\varphi\|_{0,\infty} \\ &\leq \gamma_2(h)\|T\varphi\|_{2,\infty} + C\|T\varphi\|_2 \leq \gamma_2(h)\|\varphi\|_{C^{\epsilon_1}(\Omega)} + C\|\varphi\| \\ &\leq (C\gamma_2(h)h^{-3/2-\epsilon_1} + C)\|\varphi\| \leq C\|\varphi\| \end{aligned}$$

where $0 < \epsilon_1 \leq \min\left(\frac{1}{2} - \eta, \delta\right)$ and $\varphi \in S_h$.

If we define $u_h(t) \in S_h$ via (3.2) with some choice of $v_h \in S_h$ as usual, then (4.1) shows that if $0 < t \leq \tau$,

$$\|u(t) - u_h(t)\|_{0,\infty} \leq \|(T - T_h)u_t\|_{0,\infty} + \|T_h P e^{(1)}\|_{0,\infty} \tag{6.3}$$

$$\leq \gamma_r(h)\|u(t)\|_{r,\infty} + C\|e^{(1)}(t)\| \quad .$$

We can now use the results of Section IV to further analyze (6.3). For instance, if we have Condition B_h and we set $v_h = Pv$, we find that

$$\|u(t) - u_h(t)\| \leq C(t_0)(\gamma_r(h) + h^r)\|v\| \quad . \tag{6.4}$$

for $0 < t_0 \leq t \leq \tau$. Other results can also be formulated for sufficiently smooth and compatible v.

Similar estimates can be done in the interior of Ω if the appropriate estimates are known for the $\{T_h(t)\}$ family.

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In this paper, we study certain semidiscrete methods for approximating the solutions of initial boundary value problems, with homogeneous boundary conditions, for certain kinds of parabolic equations. These semidiscrete methods are based upon the availability of several different Galerkin-type approximation methods for the associated elliptic steady-state problem. The properties required of the spacial discretization methods are listed and estimates of the error made by the resulting semidiscrete approximations and of its time derivatives are given. In particular, estimates are given that require only weak

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ABSTRACT (Continued)

smoothness assumptions on the initial data. Verifications of the required properties for various Galerkin-type methods are also given.